# Continuity

We must prove 3 conditions for a function to be continuous at a specific point (x = c). And the conditions are as follows:

1) f(c) must be defined.
2) $\lim_{x \to c} f(x)$ must exist.
3) Conditions 1 and 2 must be equal, ie, $f(c) = \lim_{x \to c} f(x)$.

Example: $\dfrac{x - 1}{x^2 + x - 2}$

← Here, the denominator cannot equal to zero. So, set the denominator equal to zero and find the numbers that make the equation equal to zero. Those numbers are not within the domain.

1) Use domain rules to find out what points to test for continuity.

$$x^2 + x - 2 \neq 0$$

$$(x + 2)(x - 1) \neq 0$$

$x \neq -2$ and $x \neq 1$ ⇐ These are two points to test for continuity.

2) Because two numbers are not within domain, f(c) is undefined in both cases, and discontinuous at both points. Now, figure out what kind of discountinuity.

- If $\lim_{x \to c} f(x) = \infty$, or $\lim_{x \to c} f(x) = -\infty$, or if the limit does not exit, then there is an asymptote at x = c.
- If $\lim_{x \to c} f(x) =$ some number (#), then there is a removable discontinuity.

$$\lim_{x \to -2} \frac{x-1}{x^2 + x - 2} = \frac{-3}{0} = -\infty.$$ Therefore there is an asymptote at x = - 2

Or....

$$\lim_{x \to 1} \frac{x-1}{x^2 + x - 2} = \frac{"0"}{0} = \text{Indeterminate}. \quad \frac{x - 1}{(x - 1)(x + 2)} = \frac{1}{x + 2} = \frac{1}{1 + 2} = \frac{1}{3}$$

Therefore, there is a removable discontinuity at x = 1.

**Situation for removable discontinuity**:

1) f(c) defined
2) $\lim_{x\to c} f(x)$ exists.
3) $f(c) \neq \lim_{x\to c} f(x)$

- This usually occurs in piecewise functions:
  - $f(x) = \begin{cases} -3 & x < 2 \\ x^2-4x+1 & x > 2 \\ 1 & x = 2 \end{cases}$

**Steps:**

1) Test all points at which f(x) switches. In this case x = 2.
   a) f (2) is defined because f(x) = 1. And this tells you the function is defined at x = 2.
2) Proceed with the normal steps: Take the limit and prove that the limit equals f (c). (L= f(c) )

   a) $f(x) = \begin{cases} -3 & x < 2 \\ x^2-4x+1 & x > 2 \\ 1 & x = 2 \end{cases}$

   $\lim_{x\to 2^+} x^2 - 4x + 1$

   $2^2 - 4(2) + 1 = -3$

   Use this equation because x > 2 tells you that numbers greater than 2 use this equation.

   Since this is piecewise functions, we have to do the limit from the positive and negative directions.

   $\lim_{x\to 2^-} f(x) = -3$. Since $x \leq 2$, use -3.

   Since, the postive and negative direction limits are equal,

   $$\lim_{x\to 2^-} = \lim_{x\to 2^+}$$
   $$-3 = -3$$

   We can conclude that,

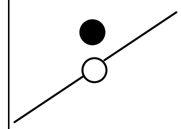
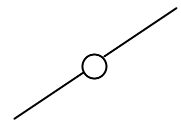
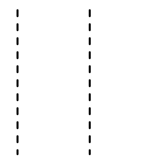
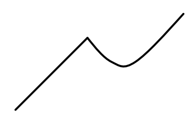
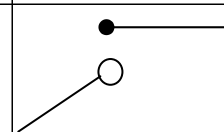
   $$\lim_{x\to 2} = -3$$

3) Now, we need to ask ourselves, does f (c) = $\lim_{x\to c} f(x)$?

   No, because f(2) = 1 and $\lim_{x\to c} f(x)$ = -3

   $$\mathbf{1 \neq -3}$$

   Therefore, it is not continuous at x = 2; It's a removable discontinuity.

| | f (c) | Limit | f(c) = Lim | Picture |
|---|---|---|---|---|
| Removable | Defined | Exists | Don't equal |  |
| Removable | Undefined | Exists | Cannot equal |  |
| Asymptote | Undefined | D.N.E (∞ or -∞) | Cannot equal |  |
| Continuous | Defined | Exists | f(c) = Lim |  |
| Jump Discontinuity (**Usually piecewise functions**) | N/A | Doesn't exist | N/A |  |