Curve Sketching

One of the most important concepts in mathematics is sketching a graph. We can sketch a graph by finding the following information:

1. Domain
2. Intercepts (x-intercepts and y-intercept)
3. Asymptotes (vertical, horizontal, or slant asymptotes)
4. Relative extrema (relative max and relative min)
5. Intervals of increasing/decreasing
6. Points of inflection
7. Intervals of concave up/down

Just think of finding the above information as a "blueprint" for a function y = f(x). We illustrate curve sketching with a couple of examples.

Example: $y = x^4 - 4x^3 + 4x^2$

1. Domain: $(-\infty,\infty)$
   Remember from your domain rules, polynomials are continuous everywhere.
2. Intercepts
   a. y-intercept: $(0,0)$ b. x-intercepts: (0,0), (2,0)

$$0 = x^4 - 4x^3 + 4x^2$$
$$0 = x^2(x^2 - 4x + 4)$$
$$0 = x^2(x-2)^2$$
$$x = 0, x = 2$$

3. Asymptotes: NONE
   Polynomials never have asymptotes. Asymptotes only occur in rational functions.

At this point, you should have this information plotted on the graph. This will help save you some time before using the first and second derivative tests.

First Derivative Test: $y' = 4x^3 - 12x^2 + 8x$

Critical numbers:

y' = 0	y' ≡ UND
$0 = 4x^3 - 12x^2 + 8x$ $0 = 4x(x^2 - 3x + 2)$ $0 = 4x(x-1)(x-2)$ $x = 0, x = 1, x = 2$	NONE. Polynomials are never undefined.

4. Relative mins: (0,0), (2,0) Relative max: (1,1)
5. Increasing Intervals: $(0,1),(2,\infty)$ Decreasing Intervals: $(-\infty,0),(1,2)$

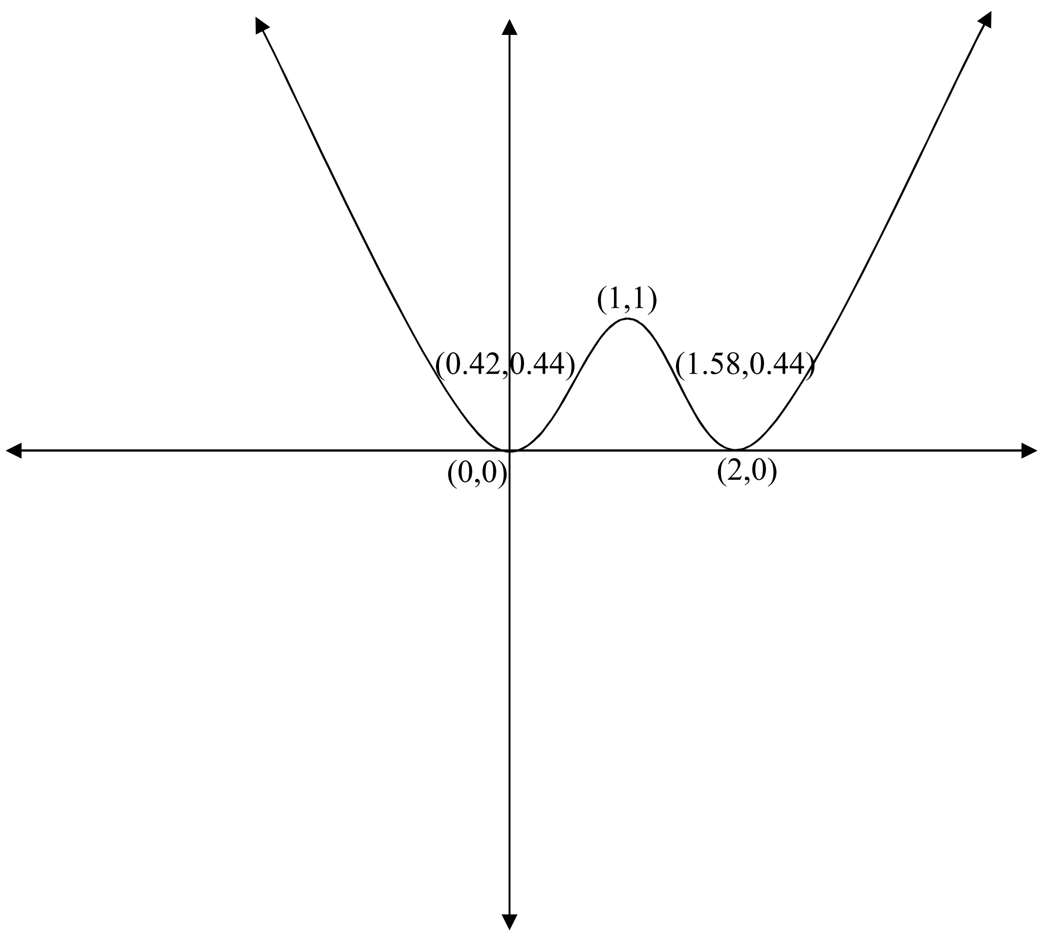
Second Derivative Test: $y'' = 12x^2 - 24x + 8$

Critical numbers:

y'' = 0	y'' ≡ UND
$0 = 12x^2 - 24x + 8$ $x = \dfrac{6 \pm \sqrt{36 - 24}}{6}$ $= \dfrac{6 \pm \sqrt{12}}{6} = \dfrac{6 \pm 2\sqrt{3}}{6} = 1 \pm \dfrac{\sqrt{3}}{3}$	NONE. Polynomials are never undefined.

6. Points of inflection: (0.42,0.44), (1.58, 0.44)
7. Concave up: (-∞,0.42), (1.58,∞) Concave down: (0.42, 1.58)

Now take the information found and sketch the graph as accurate as possible labeling all your points and asymptotes.



*Note: Make sure your graph matches your information. If it doesn't, then go back and check your work.

Example: $y = \frac{1}{x^2 - 2x}$

1. Domain: Remember in fractions, the denominator can't equal zero. So setting the denominator **NOT** equal to zero and solving gives us the following.

$$x^2 - 2x \neq 0$$
$$x(x - 2) \neq 0$$
$$x \neq 0,\ x - 2 \neq 0$$
$$x \neq 0,\ x \neq 2$$

So the domain is given by (-∞, 0) U (0, 2) U (2, ∞).

2. Intercepts: NONE
   a. y-intercept: NONE
      Setting x = 0, make the denominator undefined. So there is no intercept.
   b. x-intercepts: NONE
      The numerator is the only thing that can equal zero. But since 0 ≠ 1, then there are no x-intercepts.

3. Asymptotes:
   a. Vertical Asymptotes: Vertical asymptotes come from the denominator. So setting it equal to zero gives us

$$x^2 - 2x = 0$$
$$x(x - 2) = 0$$
$$x = 0,\ x - 2 = 0$$
$$x = 0,\ x = 2$$

Now verify that each of the above x-values found are indeed vertical asymptotes by taking one-sided limits.

$$\lim_{x \to 0^-} \frac{1}{x^2 - 2x} = \frac{1}{0} \equiv DNE = \infty \qquad \lim_{x \to 2^+} \frac{1}{x^2 - 2x} = \frac{1}{0} \equiv DNE = \infty$$

So the vertical asymptotes are x = 0 and x = 2.

   b. Horizontal Asymptotes: Horizontal asymptotes are found by taking the limit going to ±∞. Taking the limit of the above function gives us the following.

$$\lim_{x \to \pm\infty} \frac{1}{x^2 - 2x} = \frac{1}{\pm\infty} = 0$$

So the horizontal asymptote is given by y = 0. You know that your answer is correct by using precalculus. Remember, the function is "bottom heavy", so the horizontal asymptote is zero.

At this point, you should have this information plotted on the graph. This will help save you some time before using the first and second derivative tests.

First Derivative Test: $y' = \frac{-2(1-x)}{(x^2-2x)^2}$

Critical numbers:

y' = 0	y' ≡ UND
$0 = -2(1-x)$ $x = 1$	x = 0, 2 Since these are asymptotes, they can't be extrema.

4. Relative min: NONE Relative max: (1,1)
5. Increasing intervals: (-∞, 0), (0, 1) Decreasing Intervals: (1,2),(2, ∞)

Second Derivative Test: $y'' = \frac{2(-3x^2+6x-4)}{(x^2-2x)^3}$

Critical numbers:

y" = 0	y" ≡ UND
NONE	x = 0, 2 Since these are asymptotes, they can't be points of inflection.

4. Points of inflection: NONE
5. Concave up: (-∞,0), (2,∞) Concave down: (0,2)

Now take the information found and sketch the graph as accurate as possible labeling all your points and asymptotes.