Integration

F(x) take derivative and ---------→ $f'(x)$ take integral ---→ f(x), and as you can see this takes you back to your original function.

- Integration is used mainly to sum up areas, or volumes.
- Basic formula:
  - n – exponent A – coefficient
  - $\int Ax^n dx = A\int x^n dx$
  - So, on the 2nd step, the coefficient moves out front, and [dx] tells you what variable you're going to integrate.
  - Then, $\frac{x^{n+1}}{n+1}$ by adding 1 to the exponent and divide by the same thing.
  - Then, multiply back coefficient, and add C into your answer
    - $A\frac{x^{n+1}}{n+1} + C$

Example:

$$\int(4x^2 - 2x + 4)dx$$

$$4\int\frac{x^{2+1}}{2+1}dx - 2\int\frac{x^{1+1}}{1+1}dx + 4\int\frac{x^{0+1}}{0+1}$$

$$\frac{4x^3}{3} - \frac{2x^2}{2} + 4x + C$$

} Do each one separate if connected by + or –

When you have,

$$\int \# dx$$

$$\int \# \frac{x^{0+1}}{0+1}dx = \# x$$

So, memorize the $\int \# dx$ always is that #x.

Rule $\int x^n = \frac{x^{n+1}}{n+1}$ only works if the base is in the numerator, so if it is not, move it up.

Example:

$$\int(2x+4-\frac{1}{x^2}+\sqrt{x})dx$$

$$\int(2x+4-x^{-2}+(x^{\frac{1}{2}}))dx$$

$$2\int\frac{x}{1+1}^{1+1}+4\int dx-\int\frac{x}{-2+1}^{-2+1}dx+\int\frac{x}{\frac{1}{2}+\frac{2}{2}}^{\frac{1}{2}+\frac{2}{2}}dx$$

$$\frac{2x^2}{2}+4x-\frac{x}{-1}^{-1}+\frac{x}{\frac{3}{2}}^{\frac{3}{2}}+C$$

$$x^2+4X+\frac{1}{x}+\frac{2}{3}x^{\frac{3}{2}}+C$$

↓

Dividing by fraction, flip & multiply.

Rule:

$$\int e^{\#x}dx=\frac{e}{\#}^{\#X}+C$$

Example:

$$\int e^{-5x}dx=\frac{e}{-5}^{-5x}+C$$

$$\int e^{\frac{1}{4}y}dy=\frac{e}{\frac{1}{4}}^{\frac{1}{4}x}+C$$

Example:

$$4e^{\frac{1}{4}y}+C$$

Rule:

$$\int\frac{1}{x}dx=\ln|x|+C$$

$$\int \frac{2x}{x^2+1}\,dx$$

*Ask yourself is there anything I can take derivative of and get what is left in the problem (All you need in the answer of derivative is same number of variables to the same power

$u = x^2 + 1$ **(1. assign u-value)**

$\frac{du}{dx} = 2x$ **(2. solve for dx)**

$$\frac{du}{2x} = \frac{2x\,dx}{2x}$$

$$\frac{du}{2x} = dx$$

3. **Substitute** – whatever is your substituted term is replaced by u in the same manner as it was in your original problem. So whatever happens to your x^2+1 should happen to your **u**.

$$\int \frac{2x}{u}\frac{du}{2x}$$

$\int \frac{1}{u}\,du$ ← a much easier problem- plug in what dx equals (2x stays and should be cancelled out)

4. Integrate:

$\int \frac{1}{u}\,du = \ln|u| + C$ → $\ln|x^2+1| + C$

5. plug in what u equaled to get it back in terms of x.

Ex: $\int \frac{\ln x}{x}\,dx$

1. Let u = ln x

2. $\frac{du}{dx} = \frac{1}{x}$

3. xdu =dx

4. $\int \frac{u}{x}\,x\,dx$ substitute.

5. Cancel and integrate

$$\int u\,du = \frac{u^2}{2} + C$$

6. **Plug back in what u equals**

$$\frac{(\ln x)^2}{2} + C$$

Note: When integrating logs, make sure that there x's in denominator. If so, try u-substitution; if this does not work go on to integration by parts.

Ex 3: $\int xe^{x^2}\,dx$ ← if you subtitute u=e$^{\text{something}}$ then u always has exponent.
Thus u = x^2

$$\frac{du}{dx} = 2x$$

Solve for dx : $dx\,\frac{du}{dx} = 2x\,dx$

$$\frac{du}{2x} = \frac{2xdx}{2x}$$

$$\frac{du}{2x} = dx$$

$$\int xe^u \frac{du}{2x} = \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

Trig. Ex.:

$$\int \sin 2x \cos 2x \; dx$$

1. **U = (either one) because derivative of sin is cos and cos is –sin. I'll pick sin2x to avoid negative.**
   U=sin2x
2. $\frac{du}{dx} = 2\cos 2x$ (Chain rule for trig)
3. Solve for dx
   - du = 2 cos2xdx
   - $\frac{du}{2\cos 2x} = dx$
4. **Substitute -** $\int U \cos 2x \frac{du}{2\cos 2x} = \int U \frac{du}{2} = \frac{1}{2}\int U du$
5. **Integrate**
   - $\frac{1}{2}\int U du$
   - $\frac{1}{2}\frac{U^2}{2} + C$
     $\frac{U^2}{4} + C$
6. **Plug back in U**
   - $\frac{\sin^2 2x}{4} + C$

When dealing with definite integrals, the bounds must be changed during the course of substituting in a new variable:

$$\int_{-1}^{1} x(x^2+1)^3 dx$$

$$U = x^2 + 1$$

$$du = 2xdx$$

$$\frac{du}{2x} = dx$$

$$\int_{-1}^{1} u^3 x \frac{du}{2x}$$

But, the 1 and –1 are x values.
We need to replace them with u values.
So,

- **Plug into the following equation to get u bounds.**

$u = x^2 + 1$	$u = x^2 + 1$
$u = (1)^2 + 1$	$u = (-1)^2 + 1$
$u = 2$	$u = 2$

Now,

$$\frac{1}{2}\int_{2}^{2} u^3 du$$

$$\left[\frac{1}{2}\frac{u^4}{4}\right]_2^2$$

$$\left[\frac{u^4}{8}\right]_2^2$$

You can plug new bounds directly into the above equation, and as a result,

$$\frac{2^4}{8} - \frac{2^4}{8} = 0$$

So, it JUST happened to give 0.

Note: you could just replace u by $x^2 + 1$ since we already had this equality when we first began to solve for the integral. And, this would give you the same answer as just plugging in the new u bounds directly into $\left[\frac{u^4}{8}\right]_2^2$.

Example:

$$\int_0^{\frac{\pi}{2}} \cos\frac{2x}{3}\,dx$$

1. $U = \frac{2}{3}x$
2. $du = \frac{2}{3}dx$
3. $\frac{3}{2}du = dx$
4. By using step 1, change the x-values to u-values.
   a. $\int_0^{\frac{\pi}{3}} \cos U\left(\frac{3}{2}du\right)$
5. $\frac{3}{2}\int_0^{\frac{\pi}{3}} \cos u\,du$

   $\frac{3}{2}[\sin u]_0^{\frac{\pi}{3}}$

   $\frac{3}{2}\sin\frac{\pi}{3} - \frac{3}{2}\sin 0$
6. $\frac{3}{2}\frac{\sqrt{3}}{2} - \frac{3}{2}(0)$

   $\frac{3\sqrt{3}}{4}$

Some find it easier to solve the indefinite integral (without bounds). Get an answer + C. Then drop the + C and plug in the original bounds to answer.

$$\frac{3}{2}\left[\sin\frac{2x}{3}\right]_0^{\frac{\pi}{2}}$$

$\int_0^{\frac{\pi}{2}} \cos\frac{2x}{3}\,dx$ do the exact steps as above. Then, $\frac{3}{2}\sin\frac{2*\pi}{3*2} - \frac{3}{2}\sin 0$ and this would give the same answer as in step 6.

Special Example:

$$\int \frac{x}{\sqrt{2x-1}}\,dx$$

Step 1: Let u = 2x – 1 because it is the most complex term of the integral.
Step 2: Differentiating, you get

$$du = 2\ dx.$$

Step 3: Solve for dx.

$$\frac{du}{2} = dx$$

Step 4: Substitute back into original integral.

$$\int \frac{x}{\sqrt{u}} \frac{du}{2}$$

****Notice that the x-term does not cancel out with any other terms. Therefore, you must solve for x in terms of u. Taking u = 2x –1 and solving for x, you have:***

$$u - 1 = 2x$$

$$\frac{u-1}{2} = x$$

Step 5: Substitute the value of x back into the original integrand and simplify.

$$\int \frac{\frac{u+1}{2}}{\sqrt{u}} \frac{du}{2} \rightarrow \int\left(\frac{u+1}{2} \div \sqrt{u}\right)\frac{du}{2} \rightarrow \int\left(\frac{u+1}{2} \div \frac{1}{\sqrt{u}}\right)\frac{du}{2} \rightarrow \frac{1}{4}\int \frac{u+1}{\sqrt{u}}du$$

$$\rightarrow \frac{1}{4}\left(\int \frac{u}{\sqrt{u}}du + \int \frac{1}{\sqrt{u}}du\right)$$

Step 6: Integrate each part and simplify.

$$\frac{1}{4}\left(\int u^{1/2}du + \int u^{-1/2}du\right) \rightarrow \frac{1}{4}\left(\frac{u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2}\right) + C \rightarrow \frac{1}{4}\left(\frac{2u^{3/2}}{3} + 2u^{1/2}\right) + C$$

$$= \frac{u^{3/2}}{6} + 2u^{1/2} + C$$

Step 7: To finish the problem, the solution must be in terms of x. Since u = 2x – 1, then the final answer should be:

$$\frac{(2x-1)^{3/2}}{6} + 2(2x-1)^{1/2} + C$$

****Try this before going on to other techniques for integration.***