Integration by Parts

Formula:

$$\int u\,dv = uv - \int v\,du$$

where

u → the easiest term to differentiate using ILATE,
du → the derivative of u,
dv → the terms left in the original integral, and
v → the integral of dv.

NOTE: I L A T E is used for picking the correct term(s) to be u in integration by parts.

***I**nverse Trig*
***L**ogarithms*
***A**lgebra*
***T**rig*
***E**xponentials*

Example One:

$$\int xe^{-2x}dx$$

*1. Use **ILATE** to find out which term is **u**. In this example, x has <u>more</u> precedence than e^{-2x} because x is **algebraic** and e^{-2x} is **exponential**. Therefore.*

$$u = x \qquad dv = \int e^{-2x}dx$$

$$du = dx \qquad v = -\frac{e^{-2x}}{2}$$

2. Use the information from Step 1 to plug into the formula for integration by parts.

$$\int xe^{-2x}dx = (x)\left(-\frac{e^{-2x}}{2}\right) - \int \frac{e^{-2x}}{2}dx$$

$$= -\frac{1}{2}xe^{-2x} - \frac{1}{2}\int e^{-2x}dx$$

3. Integrate the remaining part of the integral. Please note the multiplier outside the remaining integral.

$$-\frac{1}{2}\int e^{-2x}dx$$

*--Using u-substituiton, we have **u =-2x**, **du = -2 dx**, and **dx = du/-2.** Subsitution yields the following:*

$$-\frac{1}{2}\int e^{u}\left(\frac{du}{-2}\right)$$

$$\frac{1}{4}\int e^{u}\,du$$

$$\frac{1}{4}e^{u} = \frac{1}{4}e^{-2x}$$

Therefore,

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x} + C\,.$$

Example Two:

$$\int \sec^3 x\,dx$$

*1. In this example, we must let **dv = sec^{2}x dx** because it is easier to integrate than **sec x dx**.*

$$u = \sec x \qquad\qquad dv = \int \sec^2 x\,dx$$

$$du = \sec x\tan x\,dx \qquad\qquad v = \tan x$$

2. Use the information from Step 1 to plug into the formula for integration by parts.

$$\int \sec^3 x\,dx = (\sec x)(\tan x) - \int (\tan x)(\sec x\tan x\,dx)$$

$$= \sec x\tan x - \int \tan^2 x\sec x\,dx$$

3. Integrate the remaining part of the integral. Please note the multiplier outside the remaining integral.

$$-\int \tan^2 x \sec x \, dx$$

$$-\int (\sec^2 x - 1)\sec x \, dx$$

$$-\int (\sec^3 x - \sec x) dx$$

$$-\int \sec^3 x \, dx + \int \sec x \, dx$$

--Looking at the two above integrals, one of them looks like the original. When this happens, it must be added to the other side.

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

--Integrate only on the right hand side because we are trying to solve for the original integral in question.

$$2\int \sec^3 x \, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right| + C$$

Note: The integral of sec x dx is in Chapter 5 with the integration of natural logs.

--Solve for the original integral by dividing by the coefficient in front.

$$\int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{\ln\left|\sec x + \tan x\right|}{2} + C$$

Therefore,

$$.\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

Example Three:

$$\int x^3 \sin x\, dx$$

1. Use ***ILATE*** *to find out which term is* ***u****. In this example, x has* <u>*more*</u> *precedence than sin x because* x^3 *is* ***algebraic*** *and sin x is* ***trig****. Therefore,*

$$u = x^3 \qquad dv = \int \sin x\, dx$$

$$du = 3x^2\, dx \qquad v = -\cos x$$

--This integration is going to require integration by parts several times. At this point, set up a table with three columns, ***u & du, sign, and dv & v****. Keep finding du and v until you get a zero in the du column. The answer will be the direction of the arrows in the table.*

sign	u and du	Dv and v
+ →	x^3 ↘	sin x
- →	$3x^2$ ↘	-cos x
+ →	6x ↘	-sin x
- →	6 ↘	cos x
+	0	sin x

2. Combine the answers in the table above. Simplify.

$$\int x^3 \sin x\, dx = (x^3)(-\cos x) - (3x^2)(-\sin x) + (6x)(\cos x) - (6)(\sin x) + C$$

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C$$

Hence,

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C.$$

Example Four:

$$\int x \arcsin x^2\, dx$$

1. Use ***ILATE*** *to find out which term is* ***u****. In this example, arcsin* x^2 *has* <u>*more*</u> *precedence than x because x is* ***algebraic*** *and arcsin* x^2 *is* ***inverse trig****. Therefore.*

$$u = \arcsin x^2 \qquad dv = \int x\, dx$$

$$du = \frac{2x}{\sqrt{1-x^4}}\, dx \qquad v = \frac{x^2}{2}$$

2. Use the information from Step 1 to plug into the formula for integration by parts.

$$\int x \arcsin x^2\, dx = (\arcsin x^2)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{2x}{\sqrt{1-x^4}}\, dx\right)$$

$$= \frac{1}{2} x \arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}}\, dx$$

3. Integrate the remaining part of the integral. Please note the multiplier outside the remaining integral.

$$-\int \frac{x^3}{\sqrt{1-x^4}}\,dx$$

--This integral almost resembles an ***arcsin*** *form but it does not because the derivative of x^4 is $4x^3$, which is present in the integral. Using u-substituiton, we have* ***$u = 1-x^4$****,* ***$du = -4x^3\,dx$****, and* ***$dx = du/-4x^3$****. Subsitution yields the following:*

$$-\int \frac{\cancel{x^3}}{\sqrt{u}}\left(\frac{du}{-4\cancel{x^3}}\right)$$

$$\frac{1}{4}\int \frac{du}{\sqrt{u}}$$

$$\frac{1}{4}\int u^{-1/2}du$$

$$\frac{1}{4}\left(\frac{u^{1/2}}{1/2}\right)=\frac{1}{4}\left(2u^{1/2}\right)$$

$$\frac{1}{2}u^{1/2}=\frac{1}{2}\sqrt{1-x^4}$$

Therefore,

$$\int x\arcsin x^2\,dx=\frac{1}{2}x\arcsin x^2+\frac{1}{2}\sqrt{1-x^4}+C.$$