ε-δ Definition for Limits

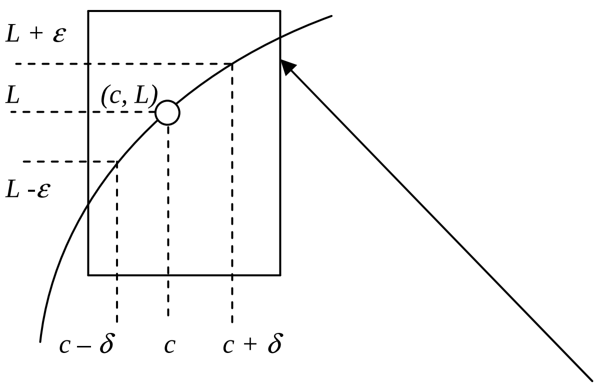
You will have to memorize the following definition for limits (a definite quiz question):

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x\text{-}c| < \delta$, then $|f(x)\text{-}L| < \varepsilon$.



This is basically saying that for some function of f(x), if you pick a point to take the limit of as x approaches c, you can pick an increment on either side of c (your δ), and there will be a corresponding increment on either side of the limit L (called ε).

Example One:

Show that

$$\lim_{x \to 2} 3x - 5 = 1$$

using the ε-δ definition for limits.

**Solution One:*

1. *Show that the above statement is true by plugging in evaluating the limit.*

$$\lim_{x \to 2} 3x - 5 = 1$$
$$3(2) - 5 = 1$$
$$6 - 5 = 1$$
$$1 = 1$$

2. *Plug in the information from the given equation into the limit definition and simplify.*

$$|f(x) - L| < \varepsilon \qquad 0 < |x\text{-}c| < \delta$$
$$|3x - 5 - 1| < \varepsilon \qquad 0 < |x\text{-}2| < \delta$$
$$|3x - 6| < \varepsilon \qquad 0 < |x\text{-}2| < \delta$$

3. Make the first equation look like the **|***x-c***|** *in the second equation.*

$$|3x-6| < \varepsilon \qquad 0 < |x\text{-}2| < \delta$$

$$|3(x-2)| < \varepsilon \qquad 0 < |x\text{-}2| < \delta$$

$$3|x-2| < \varepsilon \qquad 0 < |x\text{-}2| < \delta$$

$$|x-2| < \frac{\varepsilon}{3} \qquad 0 < |x\text{-}2| < \delta$$

4. Since they both are equivalent, then we can equate them using

$$\frac{\varepsilon}{K} = \delta.$$

In this example, K = 3; therefore, the solution is

$$\frac{\varepsilon}{3} = \delta$$

and the limit has been proven.

**Solution Two:*

1. Set up the following inequality.

$$|f(x) - L| \leq K|x-c|$$

2. Plug in the information from the given equation into the limit definition and simplify.

$$|3x-5-1| \leq K|x-2|$$

$$|3x-6| \leq K|x-2|$$

$$|3(x-2)| \leq K|x-2|$$

$$3|x-2| \leq K|x-2|$$

$$K = 3$$

3. Using

$$\frac{\varepsilon}{K} = \delta$$

for this example, the solution is

$$\frac{\varepsilon}{3} = \delta$$

and the limit has been proven.

<u>Example Two:</u>

Show that

$$\lim_{x \to 3} x^2 = 9$$

using the ε-δ definition for limits.

**Solution One:*

1. Show that the above statement is true by evaluating the limit.

$$\lim_{x \to 3} x^2 = 9$$

$$3^2 = 9$$

$$9 = 9$$

2. Plug in the information from the equation into the definition and simplify.

$$|f(x)-L|<\varepsilon \qquad 0<|x\text{-}c|<\delta$$
$$|x^2-9|<\varepsilon \qquad 0<|x\text{-}3|<\delta$$
$$|(x+3)(x-3)|<\varepsilon \qquad 0<|x\text{-}3|<\delta$$

3. Make the first equation look like the **|***x-c***|** *in the second equation. In this example, this is a quadratic, so we must prove the limit in a different way.*

$$|(\underset{\substack{\uparrow\\ \text{need to eliminate = K}}}{x+3})(x-3)|<\varepsilon \qquad 0<|\underset{\substack{\uparrow\\ \text{what is needed}}}{x\text{-}3}|<\delta$$

(1) Pick an arbitrary value for δ (usually δ = 1).

$$|x\text{-}3|<1$$

(2) Solve the inequality for x.

$$|x\text{-}3|<1$$
$$-1<x-3<1$$
$$2<x<4$$

(3) Find bounds for the expression that needed to be eliminated. In this example, we must bound the expression x+3. Thus we must add three to everything yielding:

$$2<x<4$$
$$(2+3)<x+3<(4+3)$$
$$5<x+3<7$$

(4) Pick the larger value of the inequality. This will be equal to K.

$$5<x+3<7$$
$$|x+3|<7$$

(5) Substitute the value of K back in the expression above.

$$|7(x-3)|<\varepsilon \qquad 0<|x\text{-}3|<\delta$$
$$7|x-3|<\varepsilon \qquad 0<|x\text{-}3|<\delta$$
$$|x-3|<\frac{\varepsilon}{7} \qquad 0<|x\text{-}3|<\delta$$

4. Since we pick a value for δ, we must write the answer as

$$\delta=\min\left(1,\frac{\varepsilon}{K}\right).$$

Thus, the solution is

$$\delta=\min\left(1,\frac{\varepsilon}{7}\right)$$

**Solution Two:*

1. Set up the following inequality.

$$|f(x)-L|\le K|x-c|$$

2. Plug in the information from the given equation into the limit definition and simplify. Since this is a quadratic, we prove the limit a different way. Follow the Step 3 in Solution One (boxed in above) to obtain the information.

$$|x^2 - 9| \le K|x - 3|$$

$$|(x + 3)(x - 3)| \le K|x\text{-}3|$$

3. Using

$$\delta = \min\left(1, \frac{\varepsilon}{K}\right)$$

for this example (we picked a arbitrary value for δ *and is a quadratic), the solution is*

$$\delta = \min\left(1, \frac{\varepsilon}{7}\right)$$

and the limit has been proven.