Logarithmic and Exponential Functions

I. Natural Log Properites:
1. ln (ab) = ln a + ln b
2. ln (a/b) = ln a – ln b
3. ln a^b = b ln a

II. Derivative of Natural Log Function

$\frac{d}{dx}[\ln x] = \frac{1}{x}$. This is the basic rule for f(x) = ln x.

In general, $\frac{d}{dx}[\ln\square] = \frac{d\square}{\square}$. Let's apply this rule to some examples.

Examples:

→ $f(x) = \ln(3x-5)$ $\quad f'(x) = \frac{3}{3x-5}$

→ $f(x) = \ln(x^2 - x + 5)$ $\quad f'(x) = \frac{2x-1}{x^2 - x + 5}$

→ $f(x) = \ln|\sin x|$ $\quad f'(x) = \frac{\cos x}{\sin x} = \cot x$

Sometimes we may have to rewrite the problem involving natural log before taking the derivative. This is what makes natural log derivatives easy to handle.

Examples:

→ $f(x) = \ln(9 - x^2)^{1/2}$

Rewriting first before taking derivative gives: $f(x) = \frac{1}{2}\ln(9 - x^2)$

Now taking the derivative gives $f'(x) = \frac{1}{2}\left(\frac{-2x}{9-x^2}\right) = \frac{-x}{9-x^2}$

→ $f(x) = \ln|x\cos x|$

$f(x) = \ln|x| + \ln|\cos x|$ $\quad f'(x) = \frac{1}{x} - \frac{\sin x}{\cos x} = \frac{1}{x} - \tan x$

→ $f(x) = \ln\left|\frac{3x-5}{5x-6}\right| = \ln|3x-5| - \ln|5x-6|$

$f'(x) = \frac{3}{3x-5} - \frac{5}{5x-6}$

III. Logarithmic Differentiation

Sometimes we need to use natural logs to take the derivative of a complex function. Whenever there are products and quotients of numerous expressions in the problem, or an expression of x raised to an expression of x we use logarithmic differentiation.

Example: $y = \dfrac{(\cos x)\sqrt{4x^2-9}}{(9x-5)^{1/3}x^{12}}$

1. First taking natural log of both sides.

   $$\ln y = \ln \frac{(\cos x)\sqrt{4x^2-9}}{(9x-5)^{1/3}x^{12}}$$

2. Rewrite the RHS using the natural log properties.

   $$\ln y = \ln|\cos x| + \frac{1}{2}\ln|4x^2-9| - \frac{1}{3}\ln|9x-5| - 12\ln|x|$$

3. Take derivative of both sides with respect to x. Note the implicit derivative on the LHS.

   $$\frac{y'}{y} = \frac{-\sin x}{\cos x} + \frac{1}{2}\left(\frac{8x}{4x^2-9}\right) - \frac{1}{3}\left(\frac{9}{9x-5}\right) - 12\left(\frac{1}{x}\right)$$

4. Simplify and solve for y'.

   $$\frac{y'}{y} = -\tan x + \frac{4x}{4x^2-9} - \frac{3}{9x-5} - \frac{12}{x}$$

   $$y' = \left[-\tan x + \frac{4x}{4x^2-9} - \frac{3}{9x-5} - \frac{12}{x}\right]y$$

5. Plug in for y.

   $$y' = \left[-\tan x + \frac{4x}{4x^2-9} - \frac{3}{9x-5} - \frac{12}{x}\right]\frac{(\cos x)\sqrt{4x^2-9}}{(9x-5)^{1/3}x^{12}}$$

Example: $y = (x-1)^x$

4. First taking natural log of both sides.

   $$\ln y = \ln(x-1)^x$$

5. Rewrite the RHS using the natural log properties.

   $$\ln y = x\ln|x-1|$$

6. Take derivative of both sides with respect to x. Note the implicit derivative on the LHS.

   $$\frac{y'}{y} = (1)\left(\ln|x-1|\right) + (x)\left(\frac{1}{x-1}\right)$$

4. Simplify and solve for y'.

   $$\frac{y'}{y} = \ln|x-1| + \frac{x}{x-1}$$

$$y' = \left[\ln|x-1| + \frac{x}{x-1}\right]y$$

6. Plug in for y.

$$y' = \left[\ln|x-1| + \frac{x}{x-1}\right](x-1)^x$$

IV. Derivative of Exponential Function

$\frac{d}{dx}\left[e^x\right] = e^x$. This is the basic rule for f(x) = e^x.

In general, $\frac{d}{dx}\left[e^{\Box}\right] = e^{\Box} d\Box$. Let's apply this rule to some examples.

Examples:

→ $f(x) = e^{2x}$ $\quad$ $f'(x) = 2e^{2x}$

→ $f(x) = e^{-5x}$ $\quad$ $f'(x) = e^{-5x}$

→ $f(x) = e^{x-\tan x}$ $\quad$ $f'(x) = (1-\sec^2 x)e^{x-\tan x}$

→ $f(x) = x^2e^{-x}$ $\quad$ $f'(x) = (2x)(e^{-x}) + (x^2)(-e^{-x}) = 2xe^{-x} - x^2e^{-x}$

Don't forget to review old derivative rules and techniques. How would you take derivative of the function below?

$$\ln(xy) = e^x + e^y$$