Partial Fractions

Consider the following function f(x) in the following form:

$$f(x) = \frac{N(x)}{D(x)}$$

Where N(x) is a function of x in the numerator and D(x) is a function of x in the denominator. We call f(x) a rational function. We can only use the method of partial fractions if f(x) is "bottom heavy".

If the denominator D(x) can be factored or is already factored, then we can consider using the method of partial fractions. Consider the following cases of the factored form of D(x).

Case 1:
If D(x) is composed of ***distinct linear factors***, ie, D(x) is of the form

$$D(x) = (a_1 x \pm b_1)(a_2 x \pm b_2)(a_3 x \pm b_3)...(a_n x \pm b_n)$$

then the decomposition of f(x) is given by:

$$f(x) = \frac{N(x)}{D(x)} = \frac{A_1}{a_1 x \pm b_1} + \frac{A_2}{a_2 x \pm b_2} + \frac{A_3}{a_3 x \pm b_3} + ... + \frac{A_n}{a_n x \pm b_n}$$

Case 2:
If D(x) is composed of ***repeated linear factors***, ie, D(x) is of the form

$$D(x) = (a_1 x \pm b_1)(a_1 x \pm b_1)(a_1 x \pm b_1)...(a_1 x \pm b_1) = (a_1 x \pm b_1)^n$$

then the decomposition of f(x) is given by:

$$f(x) = \frac{N(x)}{D(x)} = \frac{A_1}{a_1 x \pm b_1} + \frac{A_2}{(a_1 x \pm b_1)^2} + \frac{A_3}{(a_1 x \pm b_1)^3} + ... + \frac{A_n}{(a_1 x \pm b_1)^n}$$

Case 3:
If D(x) is composed of ***distinct quadratic factors***, ie, D(x) is of the form

$$D(x) = (a_1 x^2 \pm b_1 \pm c_1)(a_2 x^2 \pm b_2 \pm c_2)(a_3 x^2 \pm b_3 \pm c_3)...(a_n x^2 \pm b_n \pm c_n)$$

then the decomposition of f(x) is given by:

$$f(x) = \frac{N(x)}{D(x)} = \frac{A_1 x + B_1}{a_1 x^2 \pm b_1 x \pm c_1} + \frac{A_2 x + B_2}{a_2 x^2 \pm b_2 x \pm c_2} + \frac{A_3 x + B_3}{a_3 x^2 \pm b_3 x \pm c_3} + ... + \frac{A_n x + B_n}{a_n x^2 \pm b_n x \pm c_n}$$

Case 4:
If D(x) is composed of ***repeated linear factors***, ie, D(x) is of the form

$$D(x) = (a_1 x^2 \pm b_1 \pm c_1)(a_1 x^2 \pm b_1 \pm c_1)(a_1 x^2 \pm b_1 \pm c_1)...(a_1 x^2 \pm b_1 \pm c_1)^n$$

then the decomposition of f(x) is given by:

$$f(x) = \frac{N(x)}{D(x)} = \frac{A_1 x + B_1}{a_1 x^2 \pm b_1 x \pm c_1} + \frac{A_2 x + B_2}{(a_1 x^2 \pm b_1 x \pm c_1)^2} + \frac{A_3 x + B_3}{(a_1 x^2 \pm b_1 x \pm c_1)^3} + ... + \frac{A_n x + B_n}{(a_1 x^2 \pm b_1 x \pm c_1)^n}$$

Please note that D(x) can also be composed as a combination of the cases above. Analyze D(x) to make sure that you get the correct decomposition for f(x).

Example:

$$\int \frac{3}{x^2 - x - 2}\,dx$$

1. *Factor the denominator.*

$$\int \frac{3}{x^2 - x - 2}\,dx = \int \frac{3}{(x-2)(x+1)}\,dx$$

2. *Break the integral into **partial fractions** and solve for the coefficients.*

$$\frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

--Multiply both sides by the common denominator. Simplify.

$$(x-2)(x+1)\left[\frac{3}{(x-2)(x+1)}\right] = \left[\frac{A}{x-2} + \frac{B}{x+1}\right](x-2)(x+1)$$

$$3 = A(x+1) + B(x-2)$$

--Let x equal a number that will eliminate one of the coefficients. In this example, we will let x = 2 to eliminate B to solve for A.

$$3 = A(2+1) + B(2-2)$$
$$3 = A(3) + 0$$
$$3 = 3A$$
$$1 = A$$

--Solve for the other coefficient by eliminating the coefficient from the previous step.

$$3 = A(-1+1) + B(-1-2)$$
$$3 = 0 + B(-3)$$
$$3 = -3B$$
$$-1 = B$$

3. *Integrate each partial fraction with the solved coefficients from Step 2.*

$$\int \frac{3}{(x-2)(x+1)}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{-1}{x+1}\,dx$$

*--Using u-substitution on the first partial fraction we have **u = x –2**. Then we have **du = dx**.*

$$\int \frac{1}{u}\,du = \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|x-2| + C$$

*--Using u-substitution on the second partial fraction we have **u = x +1**. Then we have **du = dx**.*

$$\int \frac{-1}{u}\,du = -\int \frac{du}{u}$$
$$= -\ln|u| + C$$
$$= -\ln|x+1| + C$$

Hence, $\int \frac{3}{x^2 - x - 2}\,dx = \ln|x-2| - \ln|x+1| + C$.

<u>Example:</u>

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx$$

1. *Factor the denominator.*

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}dx = \int \frac{5x^2 + 20x + 6}{x(x^2 + 2x + 1)}dx = \int \frac{5x^2 + 20x + 6}{x(x+1)^2}dx$$

2. *Break the integral into* ***partial fractions*** *and solve for the coefficients.*

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

--Multiply both sides by the common denominator. Simplify.

$$x(x+1)^2\left[\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\right] = \left[\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right]x(x+1)^2$$

$$5x^2 + 20x + 6 = A(x+1)^2 + Bx(x+1) + Cx$$

--Let x equal a number that will eliminate two of the coefficients. In this example, we will let x = 0 to eliminate B and C to solve for A.

$$5(0)^2 + 20(0) + 6 = A(0+1)^2 + B(0)(0+1) + C(0)$$

$$0 + 0 + 6 = A(1)^2 + 0 + 0$$

$$6 = A$$

--Solve for either of the remaining two coefficients. In this example we will let x = -1 to eliminate A and B to solve for C.

$$5(-1)^2 + 20(-1) + 6 = A(-1+1)^2 + B(-1)(-1+1) + C(-1)$$

$$5 - 20 + 6 = 0 + 0 - C$$

$$-9 = -C$$

$$9 = C$$

--To solve for the remaining coefficient, let x be an arbitrary value and substitute the values of the coefficients found above. In this example, we will let x = 1 to solve for the remaining coefficient B.

$$5(1)^2 + 20(1) + 6 = (6)(1+1)^2 + B(1)(1+1) + (9)(1)$$

$$5 + 20 + 6 = 6(4) + B(2) + 9$$

$$31 = 24 + 2B + 9$$

$$31 = 33 + 2B$$

$$-2 = 2B$$

$$-1 = B$$

3. Integrate each partial fraction with the solved coefficients from Step 2.

$$\int \frac{5x^2+20x+6}{x^3(x+1)^2}dx = \int \frac{6}{x}dx + \int \frac{-1}{x+1}dx + \int \frac{9}{(x+1)^2}dx$$

*--Using u-substitution on the first partial fraction we have **u = x**. Then we have **du = dx**.*

$$\int \frac{6}{u}du = 6\int \frac{du}{u}$$
$$= 6\ln|u| + C$$
$$= 6\ln|x| + C$$

*--Using u-substitution on the second partial fraction we have **u = x +1**. Then we have **du = dx**.*

$$\int \frac{-1}{u}du = -\int \frac{du}{u}$$
$$= -\ln|u| + C$$
$$= -\ln|x+1| + C$$

*--Using u-substitution on the thrid partial fraction we have **u = x +1**. Then we have **du = dx**.*

$$\int \frac{9}{u^2}du = 9\int \frac{du}{u^2} = 9\int u^{-2}du$$
$$= 9\left(\frac{u^{-1}}{-1}\right) + C$$
$$= 9\left(\frac{-1}{u}\right) + C$$
$$= \frac{-9}{u} + C$$
$$= \frac{-9}{x+1} + C$$

Hence,

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx = 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C$$

There are problems more challenging than the ones above. Try to integrate the fraction below:

$$\int \frac{dx}{(x^2+1)(x^2-4)}$$