Related Rates

Whenever we talk about a related rate, we are usually referring to a unit of measurement changing per unit of time. For instance, miles per hour, feet per second, radians per minute are all rates.

Example: Compute dy/dt given the following information.

$x^2 + 2xy - 3y^2 = 12$, dx/dt = 2, x = -1, y = 2

1. Take derivative of both sides with respect to t.

   Every derivative of x multiply by dx/dt and every derivative of y multiply by dy/dt.

$$2x\frac{dx}{dt} + \left(2x\frac{dy}{dt} + 2y\frac{dx}{dt}\right) - 6y\frac{dy}{dt} = 0$$

2. Plug in the given values and solve for dy/dt.

$$2x\frac{dx}{dt} + \left(2x\frac{dy}{dt} + 2y\frac{dx}{dt}\right) - 6y\frac{dy}{dt} = 0$$

$$2(-1)(2) + \left(2(-1)\frac{dy}{dt} + 2(2)(2)\right) - 6(2)\frac{dy}{dt} = 0$$

$$-2 - 2\frac{dy}{dt} + 8 - 12\frac{dy}{dt} = 0$$

$$6 - 14\frac{dy}{dt} = 0$$

$$-14\frac{dy}{dt} = 6$$

$$\frac{dy}{dt} = -\frac{6}{14} = -\frac{3}{7}$$

For word problems, they will give you a rate and will require you to find another rate at a specific time frame. The following word problems will help aid in mastering related rates.

Example: You decide to pop a pizza in the oven. You find out that the radius of the pizza is changing at a rate of 3 in/min. Find the rate at which the area of the pizza is changing when the radius of the pizza is 5 in.

1. Write down the information given in the problem and draw a picture (if necessary) labeling all parts of it.

   Let r be the radius of the pizza (circle).
   Given: dr/dt = 3in/sec (rate of change of the radius)
   and r = 5 in
   Unknown: dA/dt = ?

2. Come up with a formula relating the information given.
   Since we are dealing with the area of a circle, then we will need to use the area of a circle which is given by $A = \pi r^2$.

3. Take derivative of both sides of the formula found in step 2 with respect to t.

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

4. Plug in and solve for the unknown result.
   From step 1, we are given r and dr/dt. Plugging in and solving for dA/dt gives us

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi (5\ in)(3\ in/\min)$$

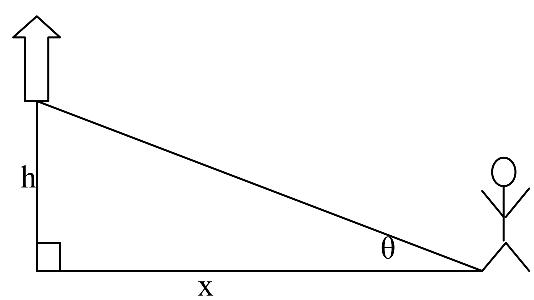
$$\frac{dA}{dt} = 30\pi\ in^2/\min$$

5. Write the answer with the correct units.
   The rate of change of the area of the pizza is at $30\pi\ in^2/min$.

Example: A boy launches his toy rocket 15 feet away with his remote control. The toy rocket gains altitude at a rate of 2.5 ft per sec. Find the rate at which the angle of elevation is changing when the rocket has gained an altitude of 15 ft.

1. Write down the information given in the problem and draw a picture (if necessary) labeling all parts of it.

   Let x be the distance between the boy and the toy rocket, h be the altitude of the rocket and θ be the angle of elevation of the rocket.
   Given: dh/dt = 2.5 ft/sec
   (rate of change of the toy rocket's altitude)
   x = 15 ft
   Unknown: dθ/dt = ?



2. Come up with a formula relating the information given.
   Since we are dealing with a right triangle with an angle, then we will need to use a formula involving a trig function. According to the picture, we are dealing with h, the side opposite the angle, and x, the side adjacent to the angle. Thus the formula that best works with this problem is given by

$$\tan\theta = \frac{h}{x}$$

   In the problem, the distance between the boy and the toy rocket is constant. So plugging in yields $\tan\theta = \frac{h}{15}$.

3. Take derivative of both sides of the formula found in step 2 with respect to t.

$$\tan\theta = \frac{h}{15}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{15}\frac{dh}{dt}$$

4. Plug in and solve for the unknown result.

   From step 1, we are given x and dh/dt. Plugging in gives us

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{15}\frac{dh}{dt}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{15}(2.5\ ft/\sec)$$

   Since we don't know θ we can still solve for it. If we use the formula we came up with, we can solve for θ by plugging in h which gives us the trig equation

$$\tan\theta = \frac{15}{15}$$

$$\tan\theta = 1$$

$$\theta = \frac{\pi}{4}$$

   Now finishing plugging in and finding dθ/dt gives us

$$\sec^2\left(\frac{\pi}{4}\right)\frac{d\theta}{dt} = \frac{1}{15}(2.5\ ft/\sec)$$

$$2\frac{d\theta}{dt} = \frac{1}{15}\left(\frac{5}{2}\ ft/\sec\right)$$

$$\frac{d\theta}{dt} = \frac{1}{12}\ rad/\sec$$

5. Write the answer with the correct units.

   The rate of change of the angle of elevation of the rocket is 1/12 rad/sec.