Trigonometric Integration

Example 1: Power of sin x is **odd** and **positive**

$$\int \sin^3 x \cos^4 x\, dx$$

1. Save **one** sin x factor.

$$\int \sin^2 x\, (\sin x) \cos^4 x\, dx$$

2. Convert as many $\sin^2 x$ terms to $\mathbf{1-\cos^2 x}$.

$$\int (1 - \cos^2 x)(\sin x) \cos^4 x\, dx$$

3. Distribute and simplify.

$$\int (\sin x \cos^4 x\, dx - \cos^6 x \sin x\, dx)$$

$$\Downarrow$$

$$\int \sin x \cos^4 x\, dx - \int \cos^6 x \sin x\, dx$$

4. Using **u-substitution**, let **u = cos x**. Then **du = -sin x dx**, and **dx = (du/-sin x)** yielding:

$$\int \sin x\, u^4 \left(\frac{du}{-\sin x}\right) - \int u^6 \sin x \left(\frac{du}{-\sin x}\right)$$

$$\int u^4(-du) - \int u^6(-du)$$

$$-\int u^4 du + \int u^6 du$$

$$-\frac{u^5}{5} + \frac{u^6}{6} + C$$

5. Now, substitute back the u expression to the answer above. Then, you would get,

$$-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$

Note: If the power of sine or cosine is even with nothing else in the problem, use the following:

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$ repeatedly.

When cosine has an odd power:
Example:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 x}{\sqrt{\sin x}} dx$$

1) Leave the odd power cosine by itself and try to convert as many **cos²x**'s into **1 – sin²x**.
2) Then, pull out a cosx from cos³x, and you would get the following:

Odd cosine power

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x \overset{\downarrow}{\cos x}}{\sqrt{\sin x}} dx$$

3) Convert **cos²x**'s into **1 – sin²x.**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1-\sin x^2)\cos x}{\sqrt{\sin x}} dx$$

4) Simplify.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{\sin^{\frac{1}{2}} x} - \frac{\sin^2 x}{\sin^{\frac{1}{2}} x} \right) \cos x dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sin^{-\frac{1}{2}} x - \sin^{\frac{3}{2}} x \right) \cos x dx$$

5) Now do u-substitution.

$u = \sin x$

$du = \cos dx$

$\frac{du}{\cos x} = dx$

$\mathrm{u} = \sin \frac{\pi}{3}$

$\mathrm{u} = \frac{\sqrt{3}}{2}$

$\mathrm{u} = \sin \frac{\pi}{6}$

$\mathrm{u} = \frac{1}{2}$

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left(u^{-\frac{1}{2}} - u^{\frac{3}{2}} \right) \cancel{\cos} x \frac{du}{\cancel{\cos} x}$$

$$\left. \frac{u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}} \right|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \Rightarrow \left. 2u^{\frac{1}{2}} - \frac{2u^{\frac{5}{2}}}{5} \right|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$\left[2\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2}} - 2\left(\frac{\frac{\sqrt{3}}{2}}{5}\right)^{\frac{5}{2}} \right] - \left[2\left(\frac{1}{2}\right)^{\frac{1}{2}} - \frac{2}{5}\left(\frac{1}{2}\right)^{\frac{5}{2}} \right] = 0.239$$

When the power of secant is even and positive:

$$\int \sec^4 3x \tan^3 3x \, dx$$

1) Leave one $\sec^2$x term and convert as many $\sec^2 x$'s to $\tan^2 x + 1$.

$$\int \sec^2 3x \; \sec^2 3x \tan^3 3x dx$$

$$\int \sec^2 3x \, (\tan^2 3x + 1) \tan^3 3x dx$$

2) Distribute and simplify.

$$\int \sec^2 3x \, (\tan^5 3x + \tan^3 3x) dx$$

3) Now, do u - substitution.

$$u = \tan 3x$$

$$du = 3\sec^2 3x dx$$

$$\frac{\mathrm{du}}{3\sec^2 3x} = dx$$

$$\int \sec^2 3x \, (\mathrm{u}^5 + \mathrm{u}^3) \frac{\mathrm{du}}{3\sec^2 3x}$$

$$\Rightarrow \quad \frac{1}{3}\int (\mathrm{u}^5 + \mathrm{u}^3) du$$

$$\Rightarrow \quad \frac{1}{3}\left[\frac{\mathrm{u}^6}{6} + \frac{\mathrm{u}^4}{4} + c\right]$$

$$\Rightarrow \quad \frac{\mathrm{u}^6}{18} + \frac{\mathrm{u}^4}{12} + c$$

$$\Rightarrow \quad \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + c$$

When the power of tan is odd and positive:

$$\int \sec^4 3x \tan^3 3x \, dx$$

1) Leave one secant and one tangent and convert the rest into secant terms.

$$\int \sec^3 3x \ (\tan^2 3x) \sec 3x \tan 3x dx$$

$$\int \sec^3 3x \, (\sec^2 3x + 1) \sec 3x \tan 3x dx$$

2) Distribute and simplify.

$$\int (\sec^5 3x + \sec^3 3x) \sec 3x \tan 3x dx$$

3) Now, do u - substitution.

$$\begin{aligned} u &= \sec 3x \\ du &= 3 \sec 3x \tan 3x dx \end{aligned}$$

$$\frac{\mathrm{du}}{3\sec 3x \tan 3x} = dx$$

$$\int \sec^5 3x + \sec^3 3\mathrm{x}) \sec 3x \tan 3x \frac{\mathrm{du}}{3\sec 3x \tan 3x}$$

$$\Rightarrow \quad \frac{1}{3}\int (\mathrm{u}^5 + \mathrm{u}^3) du$$

$$\Rightarrow \quad \frac{1}{3}\left[\frac{\mathrm{u}^6}{6} + \frac{\mathrm{u}^4}{4} + c\right]$$

$$\Rightarrow \quad \frac{\mathrm{u}^6}{18} + \frac{\mathrm{u}^4}{12} + c$$

$$\Rightarrow \quad \frac{\sec^6 3x}{18} + \frac{\sec^4 3x}{12} + c$$

Note: The previous example and this example are the same, yet we are able to use different techniques. The two answers appear to be different, but we can use trig identities to manipulate and make them the same.

When there are no secant-terms present and tangent is even and positive:

$$\int \tan^6 xdx$$

1) Take a $\tan^2$x out, and convert that $\tan^2$x into $\sec^2$x – 1.

$$\int \tan^4 x \tan^2 xdx$$

$$\int \tan^4 x(\sec^2 x - 1)dx$$

2) Distribute

$$\int \tan^4 x \sec^2 x - \tan^4 xdx$$

3) Now, we separate the integral into two parts and integrate.

$$\int \tan^4 x \sec^2 x - \int \tan^4 xdx$$

Left hand side

Right hand side

4) Now, let's work on the left hand side of the integral

$$\int \tan^4 x \sec^2 xdx$$

$$u = \tan x$$

$$du = \sec^2 xdx$$

$$\frac{du}{\sec^2 x} = dx$$

$$\int u^4 \sec^2 x \frac{du}{\sec^2 x}$$

$$\int u^4 du = \frac{u^5}{5}$$

$$\frac{\tan^5 x}{5}$$

Now, tying the RHS and the LHS together will give us our final answer

$$\boxed{\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} - \tan x + x + c}$$

5) Now, let's work on the right hand side of the integral.

$$\int \tan^4 dx \Leftarrow \text{tan is even \& positive, so repeat step 1.}$$

$$\int \tan^2 x(\tan^2 x)dx$$

$$\int \tan^2 x(\sec^2 - 1)dx$$

$$\int \tan^2 x \sec^2 x - \tan^2 xdx$$

Here we need also split the RHS integral into 2 parts. Then, we get,

$$\int \tan^2 x \sec^2 x \quad - \quad \int \tan^2 xdx$$

$$u = \tan x$$

$$du = \sec^2 xdx$$

$$\frac{du}{\sec^2 x} = dx$$

$$\int u^2 \sec^2 x \frac{du}{\sec^2 x}$$

$$\int u^2 du = \frac{u^3}{3}$$

$$\frac{\tan^3 x}{3}$$

$$\int (\sec^2 - 1)dx$$

$$\tan x - x + c$$

Tying the answers together we get:

$$\frac{\tan^3 x}{3} - \tan x + x + c$$

(Note this is just for the right side integral. We need to tie this to the LHS.

When the power of secant is odd and there are no tangent factors, use integration by parts.

$$\int \sec^3 2x\,dx$$

1. In this example, we must let ***dv = sec²x dx*** *because it is easier to integrate than* ***sec x dx****.*

$$u = \sec 2x \qquad dv = \int \sec^2 2x\,dx$$

$$du = 2\sec 2x \tan 2x\,dx \qquad v = \frac{\tan 2x}{2}$$

2. Use the information from Step 1 to plug into the formula for integration by parts.

$$\int \sec^3 2x\,dx = (\sec 2x)\left(\frac{\tan 2x}{2}\right) - \int (\frac{\tan 2x}{2})(2\sec 2x \tan 2x\,dx)$$

$$= \frac{1}{2}\sec 2x \tan 2x - \int \tan^2 2x \sec 2x\,dx$$

3. Integrate the remaining part of the integral. Please note the multiplier outside the remaining integral.

$$-\int \tan^2 2x \sec 2x\,dx$$

$$-\int (\sec^2 2x - 1)\sec 2x\,dx$$

$$-\int (\sec^3 2x - \sec 2x)dx$$

$$-\int \sec^3 2x\,dx + \int \sec 2x\,dx$$

--Looking at the two above integrals, one of them looks like the original. When this happens, it must be added to the other side.

$$\int \sec^3 2x\,dx = \frac{1}{2}\sec 2x \tan 2x - \int \sec^3 2x\,dx + \int \sec 2x\,dx$$

$$2\int \sec^3 2x\,dx = \frac{1}{2}\sec 2x \tan 2x + \int \sec 2x\,dx$$

--Integrate only on the right hand side because we are trying to solve for the original integral in question.

$$2\int \sec^3 2x\,dx = \frac{1}{2}\sec 2x \tan 2x + \frac{\ln|\sec 2x + \tan 2x|}{2} + C$$

Note: The integral of sec x dx is in Chapter 5 with the integration of natural logs.

--Solve for the original integral by dividing by the coefficient in front.

$$\int \sec^3 2x\,dx = \frac{\frac{\sec 2x \tan 2x}{2}}{2} + \frac{\frac{\ln\left|\sec 2x + \tan 2x\right|}{2}}{2} + C$$

Therefore,

$$.\int \sec^3 2x\,dx = \frac{1}{4}\sec 2x \tan 2x + \frac{1}{4}\ln\left|\sec 2x + \tan 2x\right| + C$$

When none of the rules for secant or tangent apply, convert to sines and cosines.

$$\int \csc^3 2x \tan^4 2x\,dx$$

1. Since none of the rules apply, convert to sines and cosines.

$$\int \left(\frac{1}{\sin^3 2\mathrm{x}}\right)\left(\frac{\sin^4 2x}{\cos^4 2\mathrm{x}}\right)dx$$

$$\int \frac{\sin 2x}{\cos^4 2x}dx$$

2. This integral requires u-substitution because the derivative of cos 2x is –2 sin 2x, which is in the problem. If **u = cos 2x**, then **du = -2 sin 2x dx**, and **dx = du/-2 sin 2x**. Substitution yields the following:

$$\int \frac{\sin 2x}{u^4}\left(\frac{du}{-2\sin 2x}\right)$$

$$-\frac{1}{2}\int \frac{du}{u^4}$$

$$-\frac{1}{2}\int u^{-4}du$$

$$-\frac{1}{2}\left(\frac{u^{-3}}{-3}\right)$$

$$\frac{1}{6u^3} + C = \frac{1}{6\cos^3 2x} + C = \frac{1}{6}\cdot\frac{1}{\cos^3 2x} + C = \frac{1}{6}\sec^3 2x + C$$

Hence,

$$\int \csc^3 2x \tan^4 2x\,dx = \frac{1}{6}\sec^3 2x + C.$$