Trigonometric Substitution

Basic Forms:

If you have one of the following basic forms below:

$\sqrt{a^2 - u^2}$ $\sqrt{u^2 - a^2}$ $\sqrt{u^2 + a^2}$ *or* $\sqrt{a^2 + u^2}$

Then,

$u = a\sin\theta$ $u = a\sec\theta$ $u = a\tan\theta$

and

$\sqrt{a^2 - u^2} = a\cos\theta$ $\sqrt{u^2 - a^2} = a\tan\theta$ $\sqrt{a^2 + u^2} = \sqrt{u^2 + a^2} = a\sec\theta$

*where **a** is a **constant** (number) and **u** is **an expression in terms of x**.*

Example:

$$\int \frac{dx}{x^2\sqrt{9 - x^2}}$$

1. Look at the expression under the radical and decide which of the above forms it has. In this example, the expression $9-x^2$ has the form a^2-u^2.

*2. Get the proper information to make the substitution. In this example, **u = x** and **a = 3***

-- Plug u and x into the substitutions above.

$$u = a\sin\theta \rightarrow x = 3\sin\theta$$

$$\sqrt{a^2 - u^2} = a\cos\theta \rightarrow \sqrt{9 - x^2} = 3\cos\theta$$

*-- Solve for **sin θ**.*

$$\frac{x}{3} = \sin\theta$$

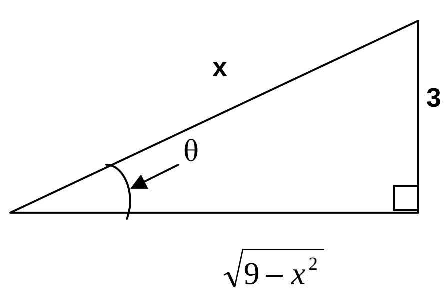
*-- Calculate **dx**. In this example, we must multiply both sides by three before finding **dx**.*

$$\frac{x}{3} = \sin\theta$$

$$x = 3\sin\theta$$

$$dx = 3\cos\theta\, d\theta$$

*--Draw a **right triangle** and label the **legs**, **hypotenuse**, and **angle θ** based on the information found for the substitution. In this example, we know that the ratio for sin θ is opposite/hypotenuse. To solve for the other side, we use the **Pythagorean Theorem**.*



$$a^2 + b^2 = c^2$$
$$a^2 + x^2 = 3^2$$
$$a^2 + x^2 = 9$$
$$a^2 = 9 - x^2$$
$$a = \sqrt{9 - x^2}$$

Note: If your calculated expression from the Pythagorean Theorem doesn't match the expression presented in the problem, then you've made a mistake. Double check your work.

3. Substitute the information found in Step 2 into the integral.

$$\int \frac{dx}{x^2\sqrt{9-x^2}} \rightarrow \int \frac{3\cos\theta\, d\theta}{(3\sin\theta)^2(3\cos\theta)}$$

4. Simplify and integrate.

$$\int \frac{3\cos\theta\, d\theta}{(3\sin\theta)^2(3\cos\theta)}$$
$$= \int \frac{d\theta}{9\sin^2\theta}$$
$$= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta}$$
$$= \frac{1}{9}\int \csc^2\theta\, \mathrm{d}\theta$$
$$= \frac{1}{9}(-\cot\theta) + C$$
$$= -\frac{1}{9}\cot\theta + C$$

5. Use the triangle in Step 2 to find an expression in terms of x. In this example, using the triangle from Step 2, we find that

$$\cot\theta = \frac{adjacent}{opposite} \longrightarrow \cot\theta = \frac{\sqrt{9-x^2}}{x}$$

--Plugging into the results yields:

$$-\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C$$

--Simplifying yields:

$$-\frac{\sqrt{9-x^2}}{9x} + C$$

Hence,

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = -\frac{\sqrt{9-x^2}}{9x} + C$$

Example:

$$\int \frac{dx}{\sqrt{4x^2+1}}$$

1. Look at the expression under the radical and decide which of the above forms it has. In this example, the expression ***$4x^2+1$****has the form* ***u^2+a^2.***

2. Get the proper information to make the substitution. In this example, ***u = 2x*** *and* ***a = 1***

-- Plug u and x into the substitutions above.

$$u = a\tan\theta \rightarrow 2x = \tan\theta$$

$$\sqrt{u^2+a^2} = a\sec\theta \rightarrow \sqrt{4x^2+1} = \sec\theta$$

-- Solve for ***tan θ****.*

$$2x = \tan\theta$$

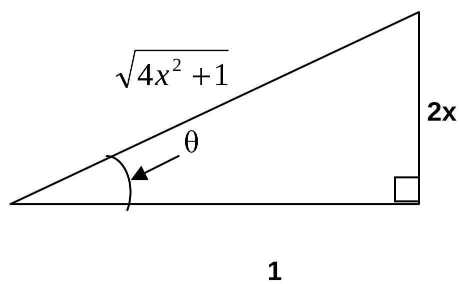
-- Calculate ***dx****. In this example, we must divide both sides by 2 before finding* ***dx****.*

$$2x = \tan\theta$$

$$x = \frac{1}{2}\tan\theta$$

$$dx = \frac{1}{2}\sec^2\theta\, \mathrm{d}\theta$$

--Draw a ***right triangle*** *and label the* ***legs****,* ***hypotenuse****, and* ***angle θ*** *based on the information found for the substitution. In this example, we know that the ratio for tan θ is opposite/adjacent. To solve for the other side, we use the* ***Pythagorean Theorem****.*



$$a^2 + b^2 = c^2$$

$$(2x)^2 + 1^2 = c^2$$

$$4x^2 + 1 = c^2$$

$$c = \sqrt{4x^2+1}$$

Note: If your calculated expression from the Pythagorean Theorem doesn't match the expression presented in the problem, then you've made a mistake. Double check your work.

3. Substitute the information found in Step 2 into the integral.

$$\int \frac{dx}{\sqrt{4x^2+1}} \rightarrow \int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\sec\theta}$$

4. Simplify and integrate.

$$\int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\sec\theta}$$

$$= \int \frac{1}{2}\sec\theta\, d\theta$$

$$= \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

5. Use the triangle in Step 2 to find an expression in terms of x. In this example, using the triangle from Step 2, we find that

$$\sec\theta = \frac{hypotenuse}{adjacent} \longrightarrow \sec\theta = \frac{\sqrt{4x^2+1}}{1} = \sqrt{4x^2+1}$$

$$\tan\theta = \frac{opposite}{adjacent} \longrightarrow \tan\theta = \frac{2x}{1} = 2x$$

--Plugging into the results yields:

$$\frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C$$

Hence,

$$\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C$$

Example:

$$\int \frac{dx}{\sqrt{x^2-9}}$$

1. Look at the expression under the radical and decide which of the above forms it has. In this example, the expression $\boldsymbol{x^2-9}$*has the form* $\boldsymbol{u^2-a^2}$.

2. Get the proper information to make the substitution. In this example, $\mathbf{u = x^2}$ *and* $\boldsymbol{a = 3}$

-- Plug u and x into the substitutions above.

$$u = a\sec\theta \rightarrow x = 3\sec\theta$$

$$\sqrt{u^2-a^2} = a\tan\theta \rightarrow \sqrt{x^2-9} = 3\tan\theta$$

-- Solve for ***sec θ****.*

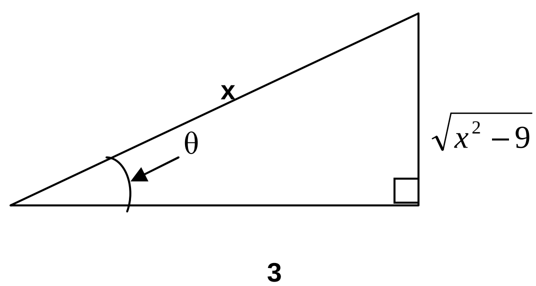
$$x = 3\sec\theta$$

--Calculate ***dx****.*

$$x = 3\sec\theta$$

$$dx = 3\sec\theta\tan\theta\,\mathrm{d}\theta$$

--Draw a ***right triangle*** *and label the* ***legs****,* ***hypotenuse****, and* ***angle θ*** *based on the information found for the substitution. In this example, we know that the ratio for sec θ is hypotenuse/adjacent. To solve for the other side, we use the* ***Pythagorean Theorem****.*



$$a^2 + b^2 = c^2$$

$$3^2 + b^2 = x^2$$

$$9 + b^2 = x^2$$

$$b^2 = x^2 - 9$$

$$b = \sqrt{x^2-9}$$

Note: If your calculated expression from the Pythagorean Theorem doesn't match the expression presented in the problem, then you've made a mistake. Double check your work.

3. Substitute the information found in Step 2 into the integral.

$$\int \frac{dx}{\sqrt{x^2-9}} \rightarrow \int \frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta}$$

4. Simplify and integrate.

$$\int \frac{3\sec\theta \tan\theta \, d\theta}{3\tan\theta}$$
$$= \int \sec\theta \, d\theta$$
$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

5. Use the triangle in Step 2 to find an expression in terms of x. In this example, using the triangle from Step 2, we find that

$$\sec\theta = \frac{hypotenuse}{adjacent} \longrightarrow \sec\theta = \frac{x}{3}$$

$$\tan\theta = \frac{opposite}{adjacent} \longrightarrow \tan\theta = \frac{\sqrt{x^2 - 9}}{3}$$

--Plugging into the results yields:

$$\ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right| + C$$

--Simplifying yields:

$$\ln\left|\frac{x + \sqrt{x^2 - 9}}{3}\right| + C$$

Hence,

$$\int \frac{dx}{\sqrt{x^2 - 9}} = \ln\left|\frac{x + \sqrt{x^2 - 9}}{3}\right| + C$$