Solutions and Initial Value Problems

Explicit Solution:
An explicit solution is a solution to an \( n \)\textsuperscript{th} order differential equation that has the dependent variable solved completely in terms of the independent variable. It is of the form

\[ y = \phi(x) \]

Implicit Solution:
An implicit solution is a solution to an \( n \)\textsuperscript{th} order differential equation that is completely in terms of both the independent variable and dependent variable. It is of the form

\[ G(x, y) = 0 \]

Example One:
Show that \( \phi(x) = 2x^3 \) is an explicit solution to \( x \frac{dy}{dx} = 3y \) on the interval \((-\infty, \infty)\).

1. Differentiate \( y = \phi(x) \) \( n \) times (\( n \equiv \) the order of the differential equation). In this example, we take the derivative of \( y = \phi(x) \) once because the order of the differential equation is one.

\[ y = \phi(x) = 2x^3 \]

\[ \frac{dy}{dx} = \phi'(x) = 6x^2 \]

2. Substitute \( \phi(x) \) and its derivatives back into the original equation and simplify.

\[ x(6x^2) = 3(2x^3) \]

\[ 6x^3 = 6x^3 \]

Since there is equality on both sides, then \( \phi(x) \) is a solution to the differential equation.

Example Two:
Show that \( y^2 + y - 3 = 0 \) is an implicit solution to \( \frac{dy}{dx} = \frac{-1}{2y} \) on the interval \((-\infty, 3)\).

1. Differentiate implicitly \( G(x, y) = 0 \).

\[ y^2 + x - 3 = 0 \]

\[ 2y \frac{dy}{dx} + 1 = 0 \]

\[ 2y \frac{dy}{dx} = -1 \]

\[ \frac{dy}{dx} = \frac{-1}{2y} \]

2. Substitute \( G(x, y) \) and its derivatives back into the original equation and simplify.

\[ \frac{-1}{2y} = \frac{-1}{2y} \]

Since there is equality on both sides, then \( G(x, y) = 0 \) is a solution to the differential equation.
Initial Value Problems:
These types of solutions occur in **initial value problems**. An initial value problem is a problem in which initial conditions is given for a differential equation. They are of the following form:

\[
F\left( x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}} \right) = 0
\]

\[
y(x_0) = y_0
\]
\[
\frac{dy}{dx}(x_0) = y_1
\]
\[\vdots\]
\[
\frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}
\]

**Example One:**
Show that \( \phi(x) = 2x^3 \) is a solution to the initial value problem

\[
x \frac{dy}{dx} = 3y \quad \quad y(2) = 16
\]
on the interval \((-\infty, \infty)\).

1. **Differentiate** \( y = \phi(x) \) \( n \) times (\( n \equiv \) the order of the differential equation). In this example, we take the derivative of \( y = \phi(x) \) once because the order of the differential equation is one.

\[
y = \phi(x) = 2x^3
\]
\[
\frac{dy}{dx} = \phi'(x) = 6x^2
\]

2. **Substitute** \( \phi(x) \) and its derivatives back into the original equation and simply.

\[
x(6x^2) = 3(2x^3)
\]
\[
6x^3 = 6x^3
\]

3. **Evaluate** \( \phi(x) \) **using the initial condition given**.

\[
16 = 2(2)^3
\]
\[
16 = 2(8)
\]
\[
16 = 16
\]

Since there is equality on both sides, then \( \phi(x) \) is a solution to the differential equation.