**Basis and Dimensions of Vector Spaces**

A set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ are said to form a **basis** if the following conditions are met.

1. The set $S$ **spans** all of their vector space.
2. The set $S$ is **linearly independent**.

To show that a set of vectors for a basis, just show that they span all of their dimension **OR** show that they are linearly independent. Both concepts imply each other because they both require the use of the determinant.

**Example:** Determine if the vectors $(2,2,2)$, $(0,0,3)$, and $(0,1,1)$ form a basis for $\mathbb{R}^3$.

→ **Spanning:** We must be able to write every vector as a linear combination of the vectors given to us. Let $\mathbf{v} = (a, b, c)$ be an arbitrary vector in $\mathbb{R}^3$. Thus, we have the following system of equations:

$$
\begin{align*}
(a, b, c) &= c_1 (2,2,2) + c_2 (0,0,3) + c_3 (0,1,1) \\
(a, b, c) &= (2c_1,2c_1,2c_1) + (0,0,3c_2) + (0,c_3,c_3) \\
(a, b, c) &= (2c_1,2c_1+c_3,2c_1+3c_2+c_3)
\end{align*}
$$

We have the following system of equations:

$$
\begin{align*}
a &= 2c_1 \\
b &= 2c_1 + c_3 \\
c &= 2c_1 + 3c_2 + c_3
\end{align*}
$$

→ **Using Gaussian Elimination,** we have the following:

$$
\begin{bmatrix}
2 & 0 & 0 \\
2 & 1 & b \\
2 & 3 & 1
\end{bmatrix}
\xrightarrow{R_3 \rightarrow R_3 - R_1}
\begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & b - a \\
0 & 3 & 1
\end{bmatrix}
\xrightarrow{R_3 \rightarrow R_3 - \frac{1}{2}R_2}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 - \frac{1}{3}R_3}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 - \frac{1}{3}R_1}
\begin{bmatrix}
1 & 0 & a/2 \\
0 & 1 & c/3 - b/3 \\
0 & 0 & b - a
\end{bmatrix}
$$

So we have the equations $c_1 = a/2$, $c_2 = c/3 - b/3$, and $c_3 = b - a$. For any value of $a$, $b$, $c$, the system always remain consistent. Therefore, the vectors span $\mathbb{R}^3$. 

Linear Independence: We must be able to write the zero vector in \( \mathbb{R}^3 \) as a linear combination of the vectors given to us and the ALL constants must be equal to zero. Thus, we have the following equation:

\[
\begin{align*}
0 &= c_1 (2,2,2) + c_2 (0,0,3) + c_3 (0,1,1) \\
0 &= (2c_1,2c_1,2c_1) + (0,0,3c_2) + (0,c_3,c_3) \\
0 &= (2c_1,2c_1 + c_3,2c_1 + 3c_2 + c_3)
\end{align*}
\]

\[
\begin{align*}
0 &= 2c_1 \\
0 &= 2c_1 + c_3 \\
0 &= 2c_1 + 3c_2 + c_3
\end{align*}
\]

\[\rightarrow\] Using Gaussian Elimination, we have the following:

\[
\begin{pmatrix}
2 & 0 & 0 \\
2 & 0 & 1 \\
2 & 3 & 1
\end{pmatrix} 
\rightarrow \begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & 1 \\
0 & 3 & 1
\end{pmatrix} 
\rightarrow \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 3 & 1
\end{pmatrix} 
\rightarrow \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

So we have the equations \( c_1 = 0, c_2 = 0, \) and \( c_3 = 0. \) Clearly, we can see that \( c_1 = c_2 = c_3 = 0. \) Therefore, the vectors are linearly independent.

\[\rightarrow\] We can also use the determinant to show that these vectors form a basis in \( \mathbb{R}^3. \) The coefficient matrix associated with the system is given by. Computing the determinant using cofactor expansion yields:

\[
\begin{vmatrix}
2 & 0 & 0 \\
2 & 0 & 1 \\
2 & 3 & 1
\end{vmatrix} = (-1)^{3+2} \begin{vmatrix}
2 & 0 \\
2 & 1
\end{vmatrix} = (-3)(2) = -6 \neq 0
\]

Therefore, the vectors form a basis for \( \mathbb{R}^3 \)
Next, we talk about the dimensions of different types of vectors. The number of vectors in any set given determines if it will be a basis for its vector space.

<table>
<thead>
<tr>
<th>Vector Object</th>
<th>Dimension of Vector Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vectors ((x_1, x_2, \ldots, x_n))</td>
<td>(n)</td>
</tr>
</tbody>
</table>
| Matrices \[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}\] | \(mn\)                     |
| Polynomials \(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n\) | \(n + 1\)                 |