**Open Sets, Closed Sets, and Boundary Points**

**Open Ball in** \( \mathbb{R}^n \):

Let \( p \) be a point in \( \mathbb{R}^n \) and \( r > 0 \). Then \( B(p, r) \), the open ball of radius \( r \) and center \( p \), is the set \( \{ y \in \mathbb{R}^n \mid |p - y| < r \} \).

**Definition of an Open Set in** \( \mathbb{R}^n \):

A subset of \( Q \) in \( \mathbb{R}^n \) is open if for every \( p \) in \( Q \), there is at least one radius \( r > 0 \) such that the open ball \( B(p, r) \) is completely contained in \( Q \).

**Examples:**

Show that the upper half-plane \( Q = \{(x, y) \mid y > 0 \} \) is open.

Let \( p = (p_1, p_2) \) be in \( Q \). Then \( p_2 > 0 \). Need to find \( r \) such that \( B(p, r) \) is completely in \( Q \). Let \( q = (q_1, q_2) \) be in \( B(p, r) \). Then \( (p_1 - q_1)^2 + (p_2 - q_2)^2 < r^2 \). Pick \( r = p_2 \). So \( (p_1 - q_1)^2 + (p_2 - q_2)^2 < p_2^2 \). Since \( (p_1 - q_1)^2 > 0 \), then \( (p_2 - q_2)^2 < p_2^2 \). So \( q_2 > 0 \). Therefore, \( Q \) is indeed open.

Show that \( Q = \{(x, y, z, w) \mid w < 2 \} \) is open.

Let \( p = (x_1, y_1, z_1, w_1) \) be in \( Q \) such that \( w_1 < 2 \). Need to find \( r \) so that \( (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2 < (w_1 - 2)^2 \). Pick \( r = |w_1 - 2| \). So since \( (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 > 0 \), it follows that \( (w_1 - w_2)^2 < (w_1 - 2)^2 \). Then \( w_2 < 2 \). Therefore, \( Q \) is indeed open.

**Properties of Open Sets:**

1. \( \mathbb{R}^n \) is open.
2. The empty set, \( \emptyset \), is open.
3. The intersection of a finite number of open sets is open.
4. The union of an arbitrary collection of open sets is open.

**Example:**

Show that \( Q = \{(x, y, z, w) \mid y > 0, w < 2 \} \)

\[Q = Q_1 \cap Q_2\]

where

\[Q_1 = \{(x, y) \mid y > 0\} \text{ and } Q_2 = \{(x, y, z, w) \mid w < 2\}\]

We have already showed that are \( Q_1 \) and \( Q_2 \) both open.

Therefore, the intersection \( Q_1 \cap Q_2 \) is open.

**Boundary Point and Boundary of a Subset:**

Let \( K \) be a subset of \( \mathbb{R}^n \) (\( K \) not need to be open). Let \( p \) be a point in \( \mathbb{R}^n \) (within or without of \( K \)). Then \( p \) is a boundary point of \( K \) if for every radius \( r > 0 \), \( B(p, r) \) contains at least one point in \( K \) and at least one point not in \( K \).

**Examples:**

Find the boundary points of \( K = [1, 2) \).

Boundary points of \( K \): \{1\} and \{2\}
Let $K$ be the punctured open disk

$$K = \{(x, y) \mid x^2 + y^2 < 1 \text{ and } (x, y) \neq (0, 0)\}.$$ 

Find the boundary of $K$.

Boundary Points: The origin and points on the circle $x^2 + y^2 = 1$.

**Closed Set:**

Definition 1: A subset of $K$ of $\mathbb{R}^n$ is closed if it contains all its boundary points. (Closure of a subset $K$ is $K$ plus all its boundary points).

Examples:

The set $[1,2)$ is not closed since $\{2\}$ is a boundary point, but it does not belong to the set $[1,2)$.

The set $K = \{(x, y) \mid x^2 + y^2 < 1 \text{ and } (x, y) \neq (1/(3^{1/2}), 1/(3^{1/2}))\}$ is not closed because the point $(1/(3^{1/2}), 1/(3^{1/2}))$ is a boundary point for $K$, but it does not belong to $K$.

Definition 2: A subset $K$ of $\mathbb{R}^n$ is closed if its compliment $\mathbb{R}^n - K = \{p \in \mathbb{R}^n \mid p \not\in K\}$ is open.

**Properties of Closed Sets:**

1. $\mathbb{R}^n$ is closed.
2. The empty set, $\emptyset$, is closed.
3. The intersection of a finite number of open sets is closed.
4. The union of an arbitrary collection of open sets is closed.