

Rational Functions and Expressions

A **rational function** is a fraction containing variables with no radicals. Examples of **rational functions** are given below.

$$\frac{1}{x} \qquad \frac{x-y}{x^2+y^2} \qquad \frac{a^2b}{a^2b+b^2}$$

There are numerous things we can do with rational functions.

Example: Find the domain of $f(x) = \frac{2x}{x^2-x-2}$.

To find the domain of $f(x) = \frac{2x}{x^2-x-2}$, we just set the denominator not equal to zero and solve.

$$\begin{aligned}x^2 - x - 2 &\neq 0 \\(x-2)(x+1) &\neq 0 \\x &\neq 2, x \neq -1\end{aligned}$$

So, the domain of this function will be given as $\{x \mid x \neq 2, x \neq -1\}$ or in set notation $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$.

Sketching graphs is another thing we can do with rational functions.

Example: Sketch the graph of $f(x) = \frac{x-2}{2x-1}$.

To sketch this graph we must make use of the following information:

- Intercepts of $f(x)$, ie, x-intercepts and y-intercept
- Asymptotes of $f(x)$, ie, vertical asymptotes and horizontal asymptotes

a. To find the y-intercept, just set $x = 0$ and solve.

$$f(0) = \frac{0-2}{2(0)-1} = \frac{-2}{-1} = 2 \qquad \text{y-intercept: } (0, 2)$$

b. To find the x-intercept(s), just set $f(x) = 0$ and solve.

$$0 = \frac{x-2}{2x-1} \rightarrow 0 = x-2 \rightarrow x = 2 \qquad \text{x-intercept: } (2, 0)$$

Remember when you set a rational function equal to zero, just look at the numerator.

c. To find the vertical asymptotes, just set the denominator = 0 and solve.

$$2x - 1 = 0 \rightarrow 2x = 1 \rightarrow x = \frac{1}{2} \qquad \text{vertical asymptote: } x = 1/2$$

d. To find horizontal asymptotes, let $x \rightarrow \pm\infty$.

$$\frac{x-2}{2x-1} \rightarrow \frac{x\left(1-\frac{2}{x}\right)}{x\left(2-\frac{1}{x}\right)} \rightarrow \frac{1-\frac{2}{x}}{2-\frac{1}{x}} \rightarrow \frac{1-0}{2-0} = \frac{1}{2} \text{ horizontal asymptote: } x = 1/2$$

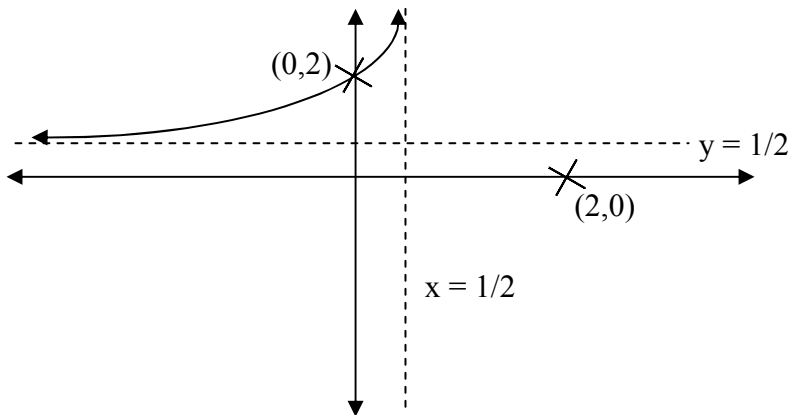
Remember, if you have $\frac{\#}{x^n}$, then that will go to zero. Another way to find horizontal asymptotes is to look at the highest powers in the top and bottom.

| Highest Power of Numerator And Denominator | Equation of Horizontal Asymptote |
|--|---|
| $m = n$ | $y = \frac{\text{coefficient of } x^m}{\text{coefficient of } x^n}$ |
| $m < n$ | $y = 0$ |
| $m > n$ | No Horizontal Asymptote (Slant Asymptote) |

So in this example, power of numerator = power of denominator; therefore, the horizontal asymptote is $y = 1/2$.

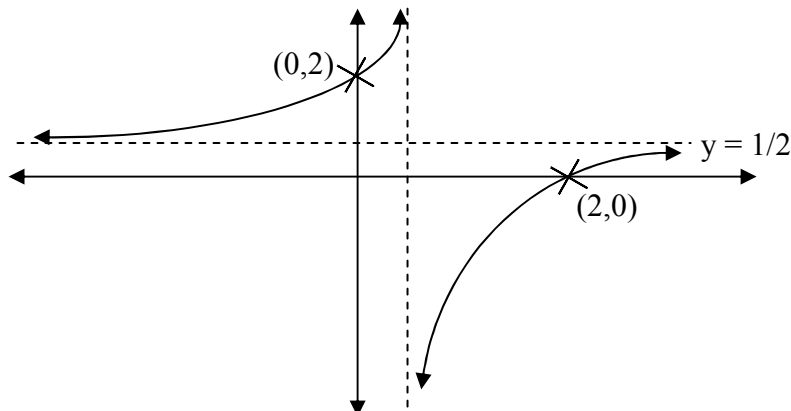
e. Use the information found above, sketch the graph.

- To find out where the curve is test on each side of the vertical asymptote. Consider $x = 0$. Then $f(0) = -1 < 0$. This means that we are above the horizontal asymptote and to the left of the vertical asymptote. So the curve would be in the top left corner.



Next, consider $x = 3$. Then $f(3) = 2 > 0$. This means that we are below the horizontal asymptote and to the right of the vertical asymptote. So the curve

would be in the bottom right corner. This is the graph of $f(x) = \frac{x-2}{2x-1}$.



Next, we will learn to add, subtract, multiply, and divide rational functions.

Example: $\frac{4a^2bc}{21r^3s} \cdot \frac{7r2s^3}{12b^2} \div \frac{a^4b^3c^2}{r^4s}$

In this example, multiply the first two fractions together by cross canceling terms.

$$\frac{\cancel{4}a^2\cancel{b}c}{\cancel{21}r^{\cancel{3}}s} \cdot \frac{\cancel{7}r^{\cancel{2}}s^{\cancel{3}}}{\cancel{12}b^{\cancel{2}}} \rightarrow \frac{a^2c}{3rs} \cdot \frac{s^2}{3b} \rightarrow \frac{a^2cs^2}{9rsb}$$

Next, divide the result by the second fraction by multiplying by the reciprocal of the second fraction.

$$\frac{a^2cs^2}{9rsb} \div \frac{a^4b^3c^2}{r^4s} \rightarrow \frac{a^2cs^2}{9rsb} \cdot \frac{r^4s}{a^4b^3c^2} \rightarrow \frac{\cancel{a}^2\cancel{s}^2}{9\cancel{r}sb} \cdot \frac{\cancel{r}^4\cancel{s}}{\cancel{a}^4\cancel{b}^3\cancel{c}^2} \rightarrow \frac{s^2}{9b} \cdot \frac{r^3}{a^2b^3c} \rightarrow \frac{r^3s^2}{9a^2b^4c}$$

Example: $\frac{1}{x-1} + \frac{2}{x+1} - \frac{3}{x^2-1}$

In this example, we need to get a common denominator because the denominators are different.

$$\frac{1}{x-1} + \frac{2}{x+1} - \frac{3}{\boxed{(x-1)(x+1)}} \quad \text{least common denominator}$$

$$\frac{x+1}{(x-1)(x+1)} + \frac{2(x-1)}{(x-1)(x+1)} - \frac{3}{(x-1)(x+1)}$$

Next, distribute and add the numerators together.

$$\frac{x+1}{(x-1)(x+1)} + \frac{2x-2}{(x-1)(x+1)} - \frac{3}{(x-1)(x+1)}$$

$$\frac{x+1+2x-2-3}{(x-1)(x+1)}$$

$$\frac{3x-4}{(x-1)(x+1)}$$

Example:
$$\frac{\frac{1}{x} - \frac{2}{x-1}}{\frac{3}{x+1}}$$

In this example, we illustrate two ways to simplify this **complex fraction**.

Solution 1:

- Getting a common denominator in the numerator yields

$$\frac{\frac{1}{x} - \frac{2}{x-1}}{\frac{3}{x+1}} = \frac{\frac{x-1}{x(x-1)} - \frac{2x}{x(x-1)}}{\frac{3}{x+1}} = \frac{\frac{-1-x}{x(x-1)}}{\frac{3}{x+1}}$$

- Dividing the fractions yields

$$\frac{-1-x}{x(x-1)} \div \frac{3}{x+1} = \frac{-1-x}{x(x-1)} \cdot \frac{x+1}{3} = \frac{-(x+1)^2}{3x(x-1)}$$

Solution 2:

- Multiplying the top and bottom by the least common denominator

$$\frac{\frac{1}{x} - \frac{2}{x-1}}{\frac{3}{x+1}} \cdot \frac{x(x-1)(x+1)}{x(x-1)(x+1)} = \frac{\frac{x(x-1)}{x} - \frac{2x(x+1)}{x-1}}{\frac{3x(x-1)(x+1)}{x+1}} = \frac{x(x-1) - 2x(x+1)}{3x(x-1)}$$

- Simplifying the top yields

$$\frac{x^2 - 1 - 2x^2 - 2x}{3x(x-1)} = \frac{-1(x^2 + 2x + 1)}{3x(x-1)}$$

We can also solve equations involving rational functions.

Example: Solve for x.
$$\frac{3}{x-1} + \frac{6}{x-2} = 4$$

(Note: In this example we know that $x = 1$ and $x = 2$ cant be solutions because they make the left hand side undefined.)

- First multiply both sides by the common denominator.

$$(x-1)(x-2) \left[\frac{3}{x-1} + \frac{6}{x-2} \right] = 4(x-1)(x-2)$$

$$3(x-2) + 6(x-1) = 4(x-1)(x-2)$$

- Simplify on both sides of the equation.

$$3x - 6 + 6x - 6 = 4(x^2 - 3x + 2)$$

$$9x - 12 = 4x^2 - 12x + 8$$

- Get equation equal to zero.

$$3x - 6 + 6x - 6 = 4(x^2 - 3x + 2)$$

$$9x - 12 = 4x^2 - 12x + 8$$

$$0 = 4x^2 - 21x + 20$$

- Solve the equation by factoring.

$$0 = 4x^2 - 21x + 20$$

$$0 = 4x^2 - 16x - 5x + 20$$

$$0 = (4x^2 - 16x) + (-5x + 20)$$

$$0 = 4x(x - 4) - 5(x - 4)$$

$$0 = (x - 4)(4x - 5)$$

$$x = 4, x = \frac{5}{4}$$

- Check the x-values in the original equation.

x = 4:

$$\frac{3}{4-1} + \frac{6}{4-2} = 4$$

$$\frac{3}{3} + \frac{6}{2} = 4$$

$$1 + 3 = 4$$

$$4 = 4$$

x = 5/4:

$$\frac{3}{\frac{5}{4}-1} + \frac{6}{\frac{5}{4}-2} = 4$$

$$\frac{3}{1/4} + \frac{6}{-3/2} = 4$$

$$12 - 4 = 4$$

$$8 \neq 4$$

Since $x = 5/4$ does not make the equation equal, we must throw it out. The only solution to the equation is $x = 4$. We call $x = 5/4$ an **extraneous** solution because it does not satisfy the equation.