<u>Roots of Polynomials</u>

A. The degree of the equation tells you the number of real roots, also called zeros, there are.

- The degree of an equation is the highest power of x.
- ***If the equation is added or subtracted, take the highest power of x.***

$y = x^4 - 2x^2 + 2$ **degree → 4**

$y = x + 2$ **degree → 1**

Should have that many zeros.

- ***If multiplied, add the powers of the x's to get the degree of the equation.***

$y = x(x^2 + 2)(x - 1) = 1 + 2 + 1 =$ **4 ← degree**

$y = (x^5 + 2)(x^2 - 4) = 5 + 2 =$ **7 ← degree**

B. **Application –**

$y = x^3 - 6x^2 - 16x$ $\Leftarrow$ by just looking, we see this equation should have 3 zeros, since the highest power of x is 3.

$$y = x(x^2 - 6x - 16)$$
$$= x(x + 2)(x - 8)$$
$$\boxed{x = 0 \quad x = -2 \quad x = 8}$$

*****Remember roots, real zeros, and x – intercepts are all the same thing.*****

$y = x^5 + 2x^4 + x^3$ $\Leftarrow$ This one should have 5 zeros.

$$y = x^3(x^2 + 2x + 1)$$
$$x^3 = 0 \quad x + 2x + 1 = 0$$
$$(x + 1)(x + 1) = 0$$
$$(x + 1)^2 = 0$$
$$(x + 1) = 0$$

$x = 0$ $\quad x = -1$ $\Leftarrow$ But, we only have 2 answers, WHY???

If you look at the power before you solved them,

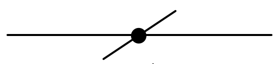
$x^3 = 0$ $(x+1)^2 = 0$

This means that x appears 3 times and x + 1 appears twice (even though there are only 2 answers.)

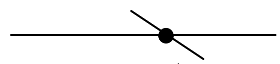
- These are called multiplicities, and they can tell you about the behavior of your function on either side of your function.

- If the power is odd, the graph crosses on either side of the root.

x^3
or
x^5

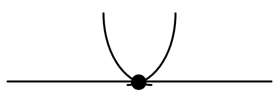

root

or

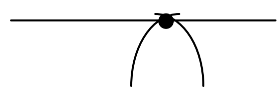

root

- If the power is even, then the graph only touches the axis.

x^2
or
x^4



or



<u>Remainder Theorem</u>

If you divide by x - #, the remainder should equal to the # plugged into the equation.

Example: $f(x) = x^3 - x^2 + 3x - 1$ / $(x - 2)$
So # = 2

$f(2) = 2^{3\,+}\,2^2 + 3(2) - 1$
$8 - 4 + 6 - 1$
$4 + 6 - 1 = 10 - 1 = 9$ ← This should be the remainder.

Lets see,

$$\begin{array}{r|rrrr} 2 & 1 & -1 & 3 & -1 \\ & & 2 & +2 & 10 \\ \hline & 1 & +1 & 5 & \boxed{9} \end{array}$$

Factor Theorem

The Factor Theorem states that if x – r is a factor, r plugged into your equation should give you zero as an answer.

$f(x) = x^3 - 23x + 10 \qquad x = -5$

$f(-5) = (-5)^3 - 23(-5) + 10$
$= -125 + 115 + 10$
$= -125 + 125 = 0 \quad \checkmark$

Rational Factor Theorem

This is helpful in factoring equations that are slightly more complex:

Example: $f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$

1) Start by finding possible roots
2) 1^{st} set are positive and negative factors of –6: $\pm 1, \ \pm 2, \ \pm 3, \ \pm 6 \ \Leftarrow$ p's
3) 2^{nd} set are positive and negative factors of coefficient of the highest power of x, and in this case the highest power of x is x^4, and it has a coefficient of 2. Then the factors are: $\pm 1, \ \pm 2 \ \Leftarrow$ q's
4) Finally we get the set of all $\dfrac{p's}{q's} \Rightarrow \pm 1, \ \pm \dfrac{1}{2}, \ \pm 2, \ \pm \dfrac{3}{2}, \ \pm 3, \ \pm 6$

You can use the factor theorem on the previous page to see which ones are toots by plugging them into the original equation. Start with the easiest ones until you find one.

$f(1) = 2(1)^4 + (1)^3 - 14(1)^2 - 19(1) - 6$
$= 2 + 1 - 14 - 19 - 6 \neq 0$

$f(-1) = 2(-1)^4 + (-1)^3 - 14(-1)^2 - 19(-1) - 6$
$= 2 + (-1) - 14 + 19 - 6 = 0$
$= -21 + 21 = 0$

Therefore, x = -1 is a root so now use synthetic division to find the rest.

$$\begin{array}{r|rrrrr} -1 & 2 & 1 & -14 & -19 & -6 \\ & & -2 & 1 & 13 & 6 \\ \hline & 2 & -1 & -13 & -6 & 0 \end{array}$$

The remainder should = 0

These are the coefficients of the new equation.

$(x+1)(2x^3 - x^2 - 13x - 6)$

$x = -1$

$(x+1)(2x^3 - x^2 - 13x - 6)$

Now, repeat with the new equation above.

We already have the p's and q's from the previous page, and we already have tried ±1, so begin with ±2.

$$f(2) = 2(2)^3 - (2)^2 - 13(2) - 6 \neq 0$$
$$f(-2) = 2(-2)^3 - (-2)^2 - 13(-2) - 6 = 0$$

So, x = -2 is a root, thus, we have (x + 2) = 0

$$\begin{array}{r|rrrr} -2 & 2 & -1 & -13 & -6 \\ & & -4 & 10 & 6 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

$(x+1)(x+2)(2x^2 - 5x - 3) \Leftarrow$ Now, we can apply normal factoring to this.

$(x+1)(x+2)(2x+1)(x-3) = 0$

$x = -1 \quad x = -2 \quad x = -\frac{1}{2} \quad x = 3$

All of these are roots to the original equation with a multiplicity, power, of 1. Thus, this tells us when power is odd the roots cross over the x-axis.

Approaching from the other way around…

Root	Mult
-1	3
2	4
5	2

Now finding the equation will follow these steps:

1st thing covert roots to factor: (x + 1)(x - 2)(x - 5)

The multiplicity tells you the power: $(x + 1)^3(x - 2)^4(x - 5)^2$

Thus, we have: $f(x) = (x + 1)^3(x - 2)^4(x - 5)^2$

*** Some common factor could have been factored out prior to the beginning, so the equation should be written as follows:

$f(x) = k(x + 1)^3(x - 2)^4(x - 5)^2$

They can then give you a condition to find what that ‘k’ value is.

f (1) = 32   so plug x = 1 and set it equal to 32.

$f(x) = k(x + 1)^3(x - 2)^4(x - 5)^2$
$32 = k(x + 1)^3(x - 2)^4(x - 5)^2$
$32 = k(1 + 1)^3(1 - 2)^4(1 - 5)^2$
$32 = k(2)^3(-1)^4(-4)^2$
32/8 = k 8/8 (1) (16)
4/16 = 16/16k
¼ = k

So, the equation obtained based on the given condition is,

$f(x) = ¼ (x + 1)^3(x - 2)^4(x - 5)^2$