① $\int \tan^8 4x \sec^6 4x \, dx$

$$\int \tan^8 4x \sec^4 4x \sec^2 4x \, dx$$

$$\int \tan^8 4x (\tan^2 4x + 1)^2 \sec^2 4x \, dx$$

$$u = \tan 4x$$
$$du = 4\sec^2 4x \, dx$$
$$\frac{du}{4} = \sec^2 4x \, dx$$

$$\frac{1}{4}\int u^8 (u^2+1)^2 \, du$$

$$\frac{1}{4}\int u^8 (u^4 + 2u^2 + 1) \, du$$

$$\frac{1}{4}\int (u^{12} + 2u^{10} + u^8) \, du$$

$$\frac{1}{4}\left[\frac{u^{13}}{13} + \frac{2u^{11}}{11} + \frac{u^9}{9}\right] + c$$

$$\frac{1}{4}\left[\frac{\tan^{13} 4x}{13} + \frac{2\tan^{11} 4x}{11} + \frac{\tan^9 4x}{9}\right] + c$$

$$\boxed{\frac{\tan^{13} 4x}{52} + \frac{\tan^{11} 4x}{22} + \frac{\tan^9 4x}{36} + c}$$

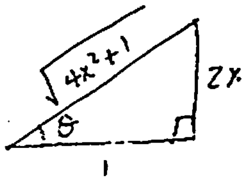
2.)

$$\int \frac{dx}{x\sqrt{4x^2+9}}$$

$u = a\tan\theta$

$u = 2x$

$a = 3$



$2x = 3\tan\theta$

$x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta\,d\theta$ and $\sqrt{4x^2+1} = \sec\theta$

$$\int \frac{1}{x\sqrt{4x^2+9}}\,dx = \int \frac{\cancel{\tfrac{3}{2}}\sec^2\theta\,d\theta}{\cancel{\tfrac{3}{2}}\tan\theta\,\sec\theta} = \int \frac{\cancel{\sec^2\theta}^{\,\sec\theta}\,d\theta}{\tan\theta\,\cancel{\sec\theta}}$$

$$= \int \frac{\sec\theta\,d\theta}{\tan\theta} = \int \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}\,d\theta$$

$$= \int \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$$\Rightarrow -\int \frac{du}{u^2} = -\int u^{-2}\,du$$

$$= \frac{\cancel{-}u^{-1}}{\cancel{-1}} + c$$

$$= \frac{1}{u} + c$$

$$= \frac{1}{\cos\theta} + c$$

$$= \sec\theta + c$$

Substitute back in for $\sec\theta$

$$\Rightarrow \boxed{\sqrt{4x^2+1} + c}$$

③ $\int x \arcsin x \, dx$

$u = \arcsin x \qquad v = \frac{x^2}{2}$

$du = \frac{1}{\sqrt{1-x^2}} \qquad dv = x$

~~Final answer~~

$$\cancel{= \boxed{\arcsin \frac{x^2}{2} - \frac{1}{2} \ln \left| \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \right| + \frac{1}{2}x + C}}$$

$$\boxed{\frac{x^2}{2} \arcsin x - \frac{1}{2} \ln \left| \frac{1+x}{\sqrt{1-x^2}} \right| + \frac{1}{2}x + C}$$

$uv - \int v \, du$

$= \arcsin x \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{1-x^2}} \, dx$

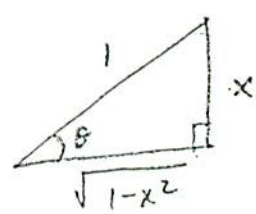
$= \arcsin x \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \, dx$

$\Downarrow$

$u = a \sin \theta$

$x = \sin \theta$

$dx = \cos \theta$



$1^2 = x^2 + b^2$

$1 = x^2 + b^2$

$b^2 = 1 - x^2$

$b = \sqrt{1-x^2}$

$\sqrt{1-x^2} = \cos \theta$

and

$x^2 = \sin^2 \theta$

$= \arcsin \frac{x^2}{2} - \frac{1}{2} \int \frac{\sin^2 \theta}{\cos \theta} \, d\theta.$

$= \arcsin \frac{x^2}{2} - \frac{1}{2} \int \frac{1 - \cos^2 \theta}{\cos \theta} \, d\theta$

$= \quad '' \quad - \frac{1}{2} \int \frac{1 \, d\theta}{\cos \theta} - \int \frac{\cos^2 \theta}{\cos \theta} \, d\theta$

$= \quad '' \quad - \frac{1}{2} \int \sec \theta \, d\theta - \int \cos \theta \, d\theta$

$= \quad '' \quad - \frac{1}{2} \ln |\sec \theta + \tan \theta| - \sin \theta + C$

$\sec \theta = \frac{1}{\sqrt{1-x^2}}$

$\tan \theta = \frac{x}{\sqrt{1-x^2}}$

4) $\int e^{2x}\sqrt{1-e^{2x}}\,dx$

$$u = 1-e^{2x}$$
$$du = -2e^{2x}\,dx$$
$$\frac{du}{-2} = e^{2x}\,dx$$

$$\frac{-1}{2}\int \sqrt{u}\,du$$

$$\frac{-1}{2}\int u^{1/2}\,du$$

$$-\frac{1}{2}\,\frac{u^{3/2}}{3/2} + C$$

$$-\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$-\frac{1}{3}u^{3/2} + C$$

$$\boxed{-\frac{1}{3}\left(1-e^{2x}\right)^{3/2} + C}$$

5) $\int \frac{216}{x^4-81}\,dx$

$$\frac{216}{(x^2-9)(x^2+9)} = \frac{216}{(x+3)(x-3)(x^2+9)}$$

$$\frac{216}{(x+3)(x-3)(x^2+9)} = \frac{A}{x+3} + \frac{B}{x-3} + \frac{Cx+D}{x^2+9}$$

$$216 = A(x-3)(x^2+9) + B(x+3)(x^2+9) + (Cx+D)(x+3)(x-3)$$

For $x = 3$, $216 = B(6)(18)$

$216 = 108B$

$\boxed{B = 2}$

For $x = -3$, $216 = A(-6)(18)$

$216 = -108A$

$\boxed{A = -2}$

For $x = 0$, $216 = (-2(-3)(9)) + (2(3)(9)) + D(3)(-3)$

$216 = 54 + 54 - 9D$

$108 = -9D$

$\boxed{D = -12}$

For $x = 1$, $216 = ((-2)(-2)(10)) + (2)(4)(10) + (C-12)(4)(-2)$

$216 = 40 + 80 + 96C$

$96 = 96C$

$\boxed{C = 1}$

$$\Rightarrow \int \frac{-2}{x+3}\,dx + \int \frac{2}{x-3}\,dx + \int \frac{x-12}{x^2+9}\,dx$$

$$= \int \frac{x-12}{x^2+9}\,dx$$

$$= \int \frac{x}{x^2+9}\,dx - \int \frac{12}{x^2+9}\,dx$$

$= \quad u = x^2+9 \quad du = 2x\,dx \quad \frac{du}{2} = x\,dx \quad - 12\int \frac{1}{x^2+9}\,dx$

$$= \frac{1}{2}\int \frac{du}{u} - 12\int \frac{1}{x^2+9}\,dx$$

$$= \frac{1}{2}\ln|x^2+9| - 12\cdot\frac{1}{3}\arctan\frac{x}{3} + C$$

$$\Rightarrow \boxed{-2\ln|x+3| + 2\ln|x-3| + \frac{1}{2}\ln|x^2+9| - 4\arctan\frac{x}{3} + C}$$

⑦ $$\int \frac{\sqrt{\ln x+3}}{x}\,dx$$

$$u = \ln x + 3$$
$$du = \frac{1}{x}\,dx$$

$$\int \sqrt{u}\,du$$

$$\int u^{1/2}\,du = \frac{2u^{3/2}}{3} + C = \boxed{\frac{2}{3}(\ln x+3)^{3/2}+C}$$

⑧ $$e^{x+y} = \arcsin e^{2y} + x^2 + y^2$$

$$\ln e^{x+y} = \ln(\arcsin e^{2y} + x^2 + y^2)$$

$$x+y = \ln(\arcsin e^{2y} + x^2 + y^2)$$

$$1 + \frac{dy}{dx} = \frac{2e^{2y}\frac{dy}{dx}}{\sqrt{1-e^{4y}}} + 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} - \frac{2e^{2y}}{\sqrt{1-e^{4x}}}\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - 1$$

$$\frac{dy}{dx}\left[1 - \frac{2e^{2y}}{\sqrt{1-e^{4x}}} - 2y\right] = 2x - 1$$

$$\boxed{\frac{dy}{dx} = \frac{2x-1}{1 - \frac{2e^{2y}}{\sqrt{1-e^{4x}}} - 2y}}$$

8. $$e^{x+y} = \arcsin e^{2y} + x^2 + y^2$$

$$e^{x+y}\left(1+\frac{dy}{dx}\right) = \frac{2e^{2y}\frac{dy}{dx}}{\sqrt{1-e^{4y}}} + 2x + 2y\frac{dy}{dx}$$

$$e^{x+y} + e^{x+y}\frac{dy}{dx} = \frac{2e^{2y}\frac{dy}{dx}}{\sqrt{1-e^{4y}}} + 2x + 2y\frac{dy}{dx}$$

$$\cancel{e^{x+y}\frac{dy}{dx} + \frac{2e^{2y}}{\sqrt{1-e^{4y}}}\frac{dy}{dx} = 2x + 2y\frac{dy}{dx} + e^{x+y}}$$

$$\cancel{\frac{dy}{dx}\left[e^{x+y} + \frac{2e^{2y}}{\sqrt{1-e^{4y}}}\right]}$$

$$e^{x+y}\frac{dy}{dx} + \frac{-2e^{2y}}{\sqrt{1-e^{4y}}}\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - e^{x+y}$$

$$\boxed{\frac{dy}{dx} = \frac{2x - e^{x+y}}{e^{x+y} - \frac{2e^{2y}}{\sqrt{1-e^{4y}}} - 2y}}$$

9 $\displaystyle\int \frac{t}{\sqrt{4-t^4}}\,dt$

$u^2 = t^4$

$u = t^2$

$du = 2t\,dt$

$\dfrac{du}{2} = t\,dt$

$\displaystyle\frac{1}{2}\int \frac{du}{\sqrt{4-u^2}}$

$\frac{1}{2}\arcsin \frac{u}{2} + c$

$\frac{1}{2}\arcsin \frac{t^2}{2} + c$