Appendix G

Elementary properties of the gamma function

G.1 Introduction

The elementary definition of the **gamma function** is **Euler's integral**:[1]

$$\boxed{\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.} \tag{G.1}$$

For the sake of convergence of the integral, it is necessary to restrict $z \in \mathbb{C}$ such that at least one of the following conditions is met: $z \in \mathbb{R}^+$, $z \notin \{\{0\} \cup \mathbb{Z}^-\}$, or $\mathrm{Im}[z] \neq 0$.

One sees instantly that

$$\forall\, n \in \mathbb{Z}^+ : \Gamma(n) = (n-1)!, \tag{G.2}$$

where, by convention, $0! = 1$. Thus the gamma function generalizes the factorial function to non-integral, negative and complex arguments, as in Fig. G.1.

From Eq. (G.1) follows the recurrence relation

$$\Gamma(z+1) = z\Gamma(z), \tag{G.3}$$

which is valid for all complex arguments z for which the integral in Eq. (G.1) converges.

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[1] In a letter to Christian Goldbach dated January 8, 1730, Euler gave the equivalent definition

$$\Gamma(x) = \int_0^1 (-\log t)^{x-1} dt$$

for a function to interpolate the factorial to non-integral arguments.

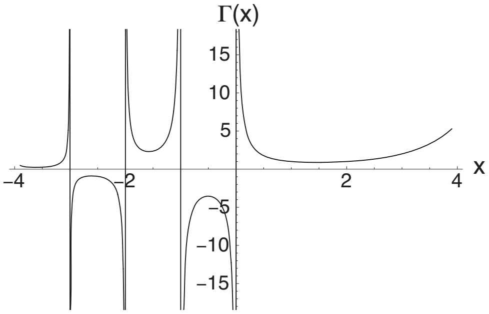

Figure G.1: Graph of $\Gamma(x)$ for $x \in (-3.9, 3.9)$.

Exercises for Section G.1

G.1.1 Derive the recurrence relation (G.3) from the integral definition of the gamma function, Eq. (G.1).

G.1.2 (a) Show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{G.4}$$

(b) Use the result of part (a) to show that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{G.5}$$

G.1.3 Show that the definition

$$\Gamma(x) = \int_0^1 (-\log t)^{x-1} dt \tag{G.6}$$

is equivalent to Eq. (G.1).

G.2 Interpolation of the factorial function

The factorial function, which is defined inductively for all natural numbers,

$$0! := 1; \quad \forall\, n \in \mathbb{Z}^+ : n! = n(n-1)!, \tag{G.7}$$

can be interpolated (in infinitely many ways!) to yield a function

$$\Gamma : \mathbb{R} \setminus \{\{0\} \cup \mathbb{Z}^-\} \mapsto \mathbb{R} \tag{G.8}$$

such that

$$\forall\, x \in \mathbb{R} \setminus \{\{0\} \cup \mathbb{Z}^-\} : \Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \Gamma(1) = 1. \tag{G.9}$$

It follows immediately from the recurrence Eq. (G.9), and the definition of the factorial function, Eq. (G.7), that every function that satisfies Eq. (G.9) interpolates the factorial function in the sense that

$$\forall\, n \in \mathbb{Z}^+ : \Gamma(n) = (n-1)!. \tag{G.10}$$

For example, the recurrence Eq. (G.9) implies that

$$\begin{gathered}\forall\, x \in (0,1] :\forall\, m \in \mathbb{Z}^+ : \\ \Gamma(x+m) = (x+m-1)(x+m-2)\cdots(x+1)x\Gamma(x)\end{gathered} \tag{G.11}$$

If the domain of *any* function $f : (0,1) \mapsto \mathbb{R}^+$, such that $f(1) = 1$, is extended from $(0,1]$ to $\mathbb{R}^+$ using Eq. (G.11), the resulting function interpolates the factorial function in the sense of Eq. (G.10).

Eq. (G.11) also defines the interpolated factorial function for all $x \in \mathbb{R} \setminus \{\{0\} \cup \mathbb{Z}^-\}$. For, if $x \in (-m, -m+1)$, then

$$\Gamma(x) = \frac{\Gamma(x+m)}{(x+m-1)(x+m-2)\cdots(x+1)x} \tag{G.12}$$

where $x+m \in (0,1)$. It follows immediately that $\Gamma(x) < 0$ for every $x \in (-2n-1, -2n)$ where $n \in \mathbb{N}$. It also follows that Γ is singular for $x = 0$ and for all negative integral values of x.

It is interesting to ask what additional condition, or conditions, on a function that interpolates the factorial function yield the gamma function as it is defined in Eq. (G.1).

One would certainly want Γ to have derivatives of all orders at points outside the set on which Γ is singular, $\{\{0\} \cup \mathbb{Z}^-\}$. In the complex plane, this implies that Γ is analytic for all $z \notin \{\{0\} \cup \mathbb{Z}^-\}$. Although this book generally avoids complex analysis, one should be aware that Eq. (G.12) and the requirement of analyticity tightly constrain a factorial-interpolating function. If the only singularities are those that are required by Eq. (G.12), then Γ is a meromorphic function. Eq. (G.12) also implies that the residue of Γ at the simple pole at $z = -m+1$ is

$$\begin{aligned}\rho_{-m+1} &= \lim_{z \to -m+1} (z+m-1) \frac{\Gamma(z+m)}{(z+m-1)(z+m-2)\cdots(z+1)z} \\ &= \frac{(-1)^{m-1}}{(m-1)!}.\end{aligned} \tag{G.13}$$

This result and Mittag-Leffler's partial-fraction theorem[2] imply that a meromorphic interpolating function with the same poles and residues can differ from the function defined in Eq. (G.1) only by an entire analytic function.

In addition to these properties, one would like an interpolated factorial function to have a qualitatively correct shape for real arguments. A function $g : \mathbb{R} \mapsto \mathbb{R}$ is called **convex** on an interval $[\alpha, \beta] \subseteq \mathbb{R}$ if and only if

$$\begin{gathered}\forall\, a, b \in [0,1] :\ni: a+b = 1 : \forall\, x_1, x_2 \in [\alpha, \beta] : \\ g(ax_1 + bx_2) \le ag(x_1) + bg(x_2).\end{gathered} \tag{G.14}$$

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[2] Konrad Knopp, *Theory of Functions, Part II*, translated by Frederick Bagemihl from the fourth German edition (New York, Dover Publications, 1947), pp. 37–38.

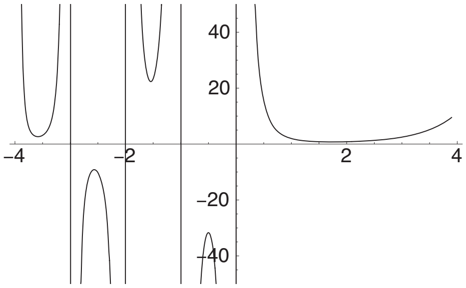

Figure G.2: Graph of $d^2\Gamma(x)/dx^2$ for $x \in (-3.9, 3.9)$.

For example, suppose that $g \in \mathcal{C}^1([\alpha,\beta];\mathbb{R})$. Then the graph of g between $x = x_1$ and $x = x_2$ must lie below the chord between $x = x_1$ and $x = x_2$, and the derivative of g must be a monotonically increasing function between $x = x_1$ and $x = x_2$, for all $x_1, x_2 \in [\alpha,\beta]$. If $g \in \mathcal{C}^2([\alpha,\beta];\mathbb{R})$, then convexity is equivalent to the condition that $d^2g/dx^2 > 0$ for all $x \in [\alpha,\beta]$. A function for which the inequality is reversed in Eq. (G.14) is called **concave**.

Figs. G.1 and G.2 illustrates that the gamma function as defined in Eq. (G.1) is convex for positive real arguments and alternately convex or concave on negative real intervals. Therefore it is not possible to require Γ to be convex.

G.2.1 Logarithmically convex functions

Bohr and Mollerup discovered[3] that the property of logarithmic convexity, along with the recurrence Eq. (G.9), is enough to define the gamma function uniquely for a real argument.

A function $g : \mathbb{R} \mapsto \mathbb{R}$ is called **logarithmically convex** on an interval $[\alpha,\beta] \subseteq \mathbb{R}$ if and only if $\log g$ is convex. If $g \in \mathcal{C}^2([\alpha,\beta];\mathbb{R})$, then logarithmic convexity on $[\alpha,\beta]$ is equivalent to the condition that $d^2(\log g)/dx^2 > 0$ for all $x \in [\alpha,\beta]$. The gamma function as defined in Eq. (G.1) is logarithmically convex, as illustrated in Fig. G.3.

G.2.2 Upper and lower bounds on the gamma function

Let Γ be a function that interpolates the factorial function in the sense defined in Eq. (G.9). It is possible to use the recurrence Eq. (G.9) and the assumption that Γ is logarithmically convex to derive inequalities between $\Gamma(x)$ and $\Gamma(x+n)$ for an arbitrary positive integer n. It then turns out that one can confine $\Gamma(x)$ between two n-dependent bounds that converge to the same limit as $n \to \infty$. The next section shows that the limiting formula gives an inifinte-product expression for the gamma function. The infinite product, in turn, is equivalent to Euler's integral, Eq. (G.1).

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[3] H. Bohr and I. Mollerup, *Loerbog I matematisk Analyse*, (Copenhagen, 1922), Vol. 3. One of the authors, Harald Bohr, was the younger brother of Niels Bohr.

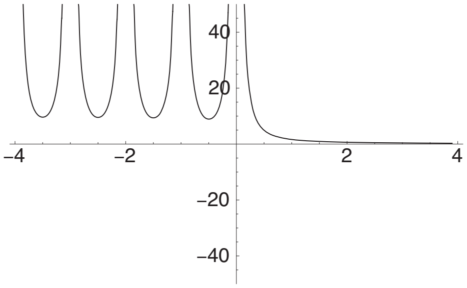

Figure G.3: Graph of $d^2(\log\Gamma(x))/dx^2$ for $x \in (-3.9, 3.9)$.

Let

$$g(x) = \log\Gamma(x), \tag{G.15}$$

where g is convex.

From Eq. (G.9),

$$g(x+1) = \log[x\Gamma(x)] = \log x + g(x). \tag{G.16}$$

This additive recurrence relation reduces the problem of finding g for $x \in (0, \infty)$ to the problem of finding g for $x \in (0, 1)$, because it follows from Eq. (G.16) by induction that

$$g(x+n) = \sum_{j=0}^{n-1} \log(x+j) + g(x). \tag{G.17}$$

It follows directly from the fact that $\Gamma(n) = (n-1)!$ for every positive integer n that

$$g(n) = \sum_{j=1}^{n-1} \log j. \tag{G.18}$$

This is a special case of Eq. (G.17) for $x = 1$ and $n \mapsto n-1$.

Now we use the convexity of g on the interval $(n, n+1)$ to obtain upper and lower bounds. Certainly

$$n + x = (1-x)n + x(n+1). \tag{G.19}$$

Then, by convexity on $(n, n+1)$, one has

$$g(n+x) \leq (1-x)g(n) + xg(n+1). \tag{G.20}$$

Substituting for $g(n)$ from Eq. (G.18), one obtains

$$g(n+x) \leq (1-x)\sum_{j=1}^{n-1} \log j + x\sum_{j=1}^{n} \log j. \tag{G.21}$$

After cancelling terms in the telescoping sum on the right-hand side, one obtains the inequality

$$g(n+x) \leq \sum_{j=1}^{n-1} \log j + x \log n. \tag{G.22}$$

Applying Eq. (G.17) yields the upper bound

$$g(x) \leq U_n(x) \tag{G.23}$$

where

$$U_n(x) := \sum_{j=1}^{n-1} \big[\log j - \log(x+j)\big] - \log x + x \log n. \tag{G.24}$$

Now, in order to obtain a lower bound, we manipulate our expressions for g such that $x+n$ is on the opposite side compared to Eq. (G.17). Since

$$n = (1-x)(x+n) + x(x+n-1), \tag{G.25}$$

the convexity of g on $(x+n-1, x+n)$ implies that for every $x \in (0,1)$,

$$g(n) \leq (1-x)g(x+n) + xg(x+n-1). \tag{G.26}$$

Substituting from Eqs. (G.17) and (G.18), one gets

$$\begin{aligned} g(n) &= \sum_{j=1}^{n-1} \log j \\ &\leq (1-x)\left[\sum_{j=0}^{n-1} \log(x+j) + g(x)\right] + x\left[\sum_{j=0}^{n-2} \log(x+j) + g(x)\right]. \end{aligned} \tag{G.27}$$

Collecting terms in x, one obtains a telescoping sum, with the result that

$$\begin{aligned} \sum_{j=1}^{n-1} \log j &\leq \sum_{j=0}^{n-1} \log(x+j) + g(x) - x\log(x+n-1) \\ &\leq \sum_{j=1}^{n-1} \log(x+j) + g(x) - x\log(x+n-1) + \log x. \end{aligned} \tag{G.28}$$

Transposing, one obtains the lower bound

$$g(x) \geq L_n(x) \tag{G.29}$$

where

$$L_n(x) := \sum_{j=1}^{n-1} \big[\log j - \log(x+j)\big] - \log x + x\log(x+n-1). \tag{G.30}$$

G.2.3 Infinite-product formula for the gamma function

The difference between the upper and lower bounds derived for $g(x) = \log \Gamma(x)$ in the previous section is

$$\begin{aligned} U_n(x) - L_n(x) &= x\big[\log n - \log(x + n - 1)\big] \\ &= x \log\left(\frac{1}{1 + \dfrac{x-1}{n}}\right), \end{aligned} \tag{G.31}$$

which approaches 0 as $n \to \infty$. Therefore, if $U_n(x)$ approaches a limit as $n \to \infty$ for every $x \in \mathbb{R}^+$, then

$$g(x) = \lim_{n\to\infty} U_n(x), \tag{G.32}$$

and $g(x)$ is uniquely defined for all $x \in \mathbb{R}^+$. This section shows that $U_n(x)$ approaches a limit, and derives an infinite-product formula for

$$\Gamma(x) = \lim_{n\to\infty} e^{U_n(x)}. \tag{G.33}$$

To show that $U_n(x)$ approaches a limit, one begins by rewriting several terms of Eq. (G.24) in terms of integrals:

$$\begin{aligned} U_n(x) &= -\sum_{j=1}^{n-1} \int_j^{j+x} \frac{dt}{t} - \log x + x \int_1^n \frac{dt}{t} \\ &= \sum_{j=1}^{n-1} \left[x \int_j^{j+1} \frac{dt}{t} - \int_j^{j+x} \frac{dt}{t} \right] - \log x. \end{aligned} \tag{G.34}$$

Fig. **??** illustrates that each term in brackets is negative. It follows that the absolute value of the sum of the bracketed terms is equal to the sum of the absolute values of the individual terms. It is possible to obtain an upper bound for the absolute value of each term, such that the sum of the upper bounds converges absolutely as $n \to \infty$. The Weierstraß M-test then guarantees absolute and uniform convergence of the functional series in the second line of Eq. (G.34) as $n \to \infty$.

In the inequality

$$|a_j - b_j| \le A - B, \tag{G.35}$$

where

$$\forall\, j : a_j \le A \quad \text{and} \quad b_j \ge B, \tag{G.36}$$

let

$$a_j = x \int_j^{j+1} \frac{dt}{t} \quad \text{and} \quad b_j = \int_j^{j+x} \frac{dt}{t}. \tag{G.37}$$

Certainly

$$\forall\, x \in (0,1): \int_j^{j+x} \frac{dt}{t} \geq \frac{x}{j+1} \quad \text{and} \quad x\int_j^{j+1} \frac{dt}{t} \leq \frac{x}{j}. \tag{G.38}$$

Then, for every $x \in (0,1)$, one obtains the upper bound

$$x\int_j^{j+1} \frac{dt}{t} - \int_j^{j+x} \frac{dt}{t} \leq \frac{x}{j} - \frac{x}{j+1} = \frac{x}{j(j+1)} \leq \frac{1}{j(j+1)} \tag{G.39}$$

for the term corresponding to j in Eq. (G.34).

The series

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = 1 \tag{G.40}$$

converges, and every term is positive. Therefore, by Eq. (G.39) and the Weierstraß M-test (Volume 1, Section 10.4.4), the series

$$\sum_{j=1}^{\infty} \left[x\int_j^{j+1} \frac{dt}{t} - \int_j^{j+x} \frac{dt}{t} \right] \tag{G.41}$$

converges uniformly on (0,1). It follows that Eq. (G.32) holds, giving us a useful formula for the logarithm of the gamma function.

Now it is easy to get an infinite product that converges to the gamma function. From Eqs. (G.24) and (G.33) one has the results

$$\Gamma(x) = \lim_{n\to\infty} \exp\left[-\log x + x\log n + \sum_{j=1}^{n-1} \big[\log j - \log(x+j)\big] \right] \tag{G.42}$$

and

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x (n-1)!}{x(x+1)\cdots(x+n-1)}. \tag{G.43}$$

Because $\lim_{n\to\infty} \frac{n}{x+n} = 1$, one can rewrite the infinite product as

$$\boxed{\Gamma(x) = \lim_{n\to\infty} \frac{n!}{x(x+1)\cdots(x+n)} n^x.} \tag{G.44}$$

Confusingly, this is often called **Euler's formula** for the gamma function. Because Gauss used this equation to define $\Gamma(z)$ for all $z \notin \{\{0\} \cup \mathbb{Z}^-\}$, a more appropriate name may be the **Gaussian definition** of the gamma function.

To show that Eqs. (G.1) and (G.44) are equivalent, one can show that the function

$$G(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dz, \quad \text{where} \quad n \in \mathbb{Z}^+, \tag{G.45}$$

reduces to the function defined in Eq. (G.1) in the limit $n \to \infty$, and that G also reduces to the Euler infinite product, Eq. (G.44), with the substitution $u = t/n$ followed by n integrations by parts and the limit $n \to \infty$.[4]

Exercises for Section G.2

G.2.1 Prove Eq. (G.40).

G.2.2 Use Eq. (G.45) to show that the function defined in Eq. (G.44) is equal to the function defined in Eq. (G.1).

G.2.3 Use Eqs. (G.3) and (G.44) to show that

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{\sin(\pi z)}. \tag{G.46}$$

You may assume without proof that

$$\frac{\sin(\pi z)}{\pi z} = \prod_{s=0}^{\infty} \left(1 - \frac{z^2}{s^2}\right). \tag{G.47}$$

G.2.4 Use Eqs. (G.3) and (G.46) to show that

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{\pi}\sin(\pi z). \tag{G.48}$$

The relation

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}} \tag{G.49}$$

is known as the **complement formula** or the **reflection formula** for the gamma function.

G.2.5 Use Eq. (G.44) to establish the **Weierstraß infinite product**

$$\boxed{\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.} \tag{G.50}$$

You may assume without proof that

$$\lim_{n\to\infty} \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n} - \log n\right) = \gamma \tag{G.51}$$

where $\gamma \approx 0.5772156649015328606$ is the **Euler-Mascheroni constant**.

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[4] George Arfken, *Mathematical Methods for Physicists*, Third Edition (New York, Academic Press, 1985), pp. 540–541.