1. Find three probability measures $\mu_1, \mu_2, \mu_3$ on $(\mathbb{R}, B_{\mathbb{R}})$ such that
\[
\mu_1 + \mu_2 \leq \mu_3, \\
\mu_2 + \mu_3 \leq \mu_1,
\]
and
\[
\mu_3 + \mu_1 \leq \mu_2.
\]
[Note: $\mu_1$ being a probability measure means $\mu_1(\mathbb{R}) = 1$.]

2. Let $\mathcal{F}$ be a $\sigma$-algebra on $X$ and let $\{\mu_n\}_{n \geq 1}$ be a family of measures on $\mathcal{F}$ such that
\[
\mu_n(A) \leq \mu_{n+1}(A) \quad \text{for all } A \in \mathcal{F}.
\]
Does the formula
\[
\mu(A) = \lim_{n \to \infty} \mu_n(A), \quad A \in \mathcal{F},
\]
define a measure on $\mathcal{F}$?

3. Is a totally bounded metric space sequentially compact? Is a sequentially compact metric space totally bounded? Justify your answers.
REAL ANALYSIS. II, SPRING 2005

Solve all. Closed book.

1. Let \( T = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \), \( a \neq b \). Show that for any polynomial \( p \),

\[
p(T) = \begin{bmatrix} p(a) & c \frac{p(a) - p(b)}{a - b} \\ 0 & \frac{p(b)}{p(b)} \end{bmatrix}.
\]

2. Let \( h_1, h_2, \ldots, h_n \) be elements of a Hilbert space \( \mathcal{H} \). Consider the Gram matrix

\[
G(h_1, h_2, \ldots, h_n) = \begin{bmatrix}
\langle h_1, h_1 \rangle & \langle h_2, h_1 \rangle & \ldots & \langle h_n, h_1 \rangle \\
\langle h_1, h_2 \rangle & \langle h_2, h_2 \rangle & \ldots & \langle h_n, h_2 \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle h_1, h_n \rangle & \langle h_2, h_n \rangle & \ldots & \langle h_n, h_n \rangle
\end{bmatrix}.
\]

Show that \( h_1, h_2, \ldots, h_n \) are linearly independent if and only if \( G(h_1, h_2, \ldots, h_n) \neq 0 \).

3. Let \( \mathcal{E} \subset \mathcal{P}(X) \) be an algebra. Also let \( C: \mathcal{E} \to [0,1] \) have the property:

\[
C(Z_1 \cup Z_2) = C(Z_1) + C(Z_2)
\]

for \( Z_1, Z_2 \in \mathcal{E} \) and \( Z_1 \cap Z_2 = \emptyset \). Show that the following statements are equivalent:

(a) For arbitrary sets \( Z, Z_1, Z_2, \ldots \) in \( \mathcal{E} \) with \( Z \subset \bigcup_{n \geq 1} Z_n \), we have

\[
C(Z) \leq \sum_{n \geq 1} C(Z_n).
\]

(b) For every decreasing sequence \( \{Z_n\}_{n \geq 1} \subset \mathcal{E} \) with \( \cap_{n \geq 1} Z_n = \emptyset \), we have

\[
\lim_{n \to \infty} C(Z_n) = 0.
\]
1. Define the successive approximate solutions $\phi_k$ of $x' = f(t, x)$ with $\phi(\tau) = \xi$? If $f \in C, Lip$ on $\mathbb{R}$, then give a detailed proof that $\phi_k$'s converge uniformly to a unique solution $\phi$ of the above IVP.

2. (a) Does a solution to $x' = (t + x)^{1/5}$ with $x(0) = 1$ exist for values of $t$ in the neighborhood of zero?

(b) If the solution to above IVP exit, will it be unique? Please justify all your statements.

(c) Write the equations satisfied by the successive approximate solutions, $\phi_k(t)$, of this IVP and compute its first two approximate solutions, $\phi_0(t)$ & $\phi_1(t)$.

3. Suppose for a given continuous function $f(t)$ the equation

$$x' = \left(\begin{array}{cc} -5 & 2 \\ -4 & 1 \end{array}\right)x + f(t)$$

has at least one solution $\phi_p(t)$ which satisfies

$$\sup\{|\phi(t)| : \tau \leq t < \infty\} < \infty.$$  

Show that all the solutions of above ODE satisfy this boundedness condition.

4. (a) Given $x' = A(t)x$ with $A(t)$ a continuous periodic function with period $T$. Show that if $\Phi(t)$ is a fundamental matrix solution for $x' = A(t)x$ then there exist a nonsingular matrix $P$ which is periodic with period $T$ and a constant matrix $R$, such that $\Phi(t) = P(t)e^{tR}$.

(b) for the differential equation $x'' = x' \cos(t)$, find the associated periodic matrix $P$ and the constant matrix $R$. **Hint:** Make use of information given in part (a).
ALGEBRA - QUALIFYING EXAMINATIONS
April 13th, 2005
V. Ramakrishna

- You must show all work to receive full credit. If all work is not shown you may not get any points.

- **CHOICE:** Do any FIVE of the questions below. Indicate on this page which Qs you do not want graded.

- Please write your name on every sheet of paper you use.

- You may use both sides of a page and also attach additional sheets if necessary.

- Write your answers only in the space provided, i.e., not on margins, not in the space separating two parts of a question. Further, **CLEARLY** separate answers to each subpart.
Q1) Show that $\frac{R}{I}$ is a field iff $I$ is a maximal ideal in $R$.

Q2) Let $p$ be a prime and let $G$ be a group with cardinality $p^2$. Show $G$ is abelian (State precisely any other results you use - but these need not be proved).

Q3) Show that every finite integral domain is a field.

Q4) Let $Z(G)$ denote the centre of a finite group $G$. Explain why it is a normal subgroup. Show, further, that if the quotient group $\frac{G}{Z(G)}$ is cyclic, then $G$ is abelian.

Q5) Define i) A term order; ii) Define the lex term order; iii) Define a Groebner basis for an ideal; iv) Define a reduced Groebner basis for an ideal.

Q6) State Hilbert Nullstellensatz. Explain how this plus a Groebner basis calculation can decide if a system of polynomial equations possesses a solution.

Q7) Consider the system of equations and inequalities: $f_1 = 0, \ldots, f_s = 0, g_1 \neq 0, \ldots, g_t \neq 0$, where the $f_i, g_k$ are polynomials in $C[x_1, \ldots, x_n]$. Show how to rewrite this as a system of equations alone (Hint: Use the Rabinowitch trick).

Q8) Define a Noetherian ring. State two other conditions which are equivalent to your definition.

Q9) Show that a PID is a Noetherian ring.
Qualifying Examination in Complex Analysis

Closed Book

April 15, 2005

Write your name on each page
Answer as many questions as you can.

Give the definition of the derivative of the complex function

\[ w = f(z). \]

What is the reason, that this definition is more restrictive than the corresponding one for a real valued function of a real variable?

Why is the function \( f(z) = \bar{z} \) nowhere differentiable?

Assume that the function

\[ w = f(z) = u(x,y) + iv(x,y) \]

is differentiable in a region \( G \). Derive the Cauchy-Riemann equations.

Consider Cauchy's fundamental theorem:
If \( f(z) \) is differentiable in a simply connected region \( G \), then

\[ \int_C f(z) \, dz = 0, \]

where \( C \) is an arbitrary closed path lying in \( G \).

Prove this theorem under the assumption that the partial derivatives of the functions \( u \) and \( v \) are continuous. Hint: use Green's theorem

\[ \int_{\partial R} P \, dx + Q \, dy = \int_R (Q_x - P_y) \, dx \, dy. \]

E. Goursat proved the Cauchy theorem without the assumption of continuity of the derivatives. The great significance of Goursat's theorem warrants to call the fundamental theorem Cauchy-Goursat theorem. Could you reconstruct Goursat's proof?

We now come to the most important consequence of the Cauchy-Goursat theorem: State and prove the Cauchy integral formula for \( f(z) \), which is differentiable throughout the region \( G \).

Give the integral formula for \( f^{(n)}(z) \) as well.

The function of the form

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n, \]
where the series is convergent for every value of $z$, is called entire function. Under the assumption that $f(z)$ is bounded:

$$|f(z)| \leq M,$$

and using the integral formula for the derivatives, it is easy to deduce the Cauchy inequality (prove it)

$$|a_n| \leq \frac{M}{R^n} \quad \text{for} \quad n = 1, 2, \ldots, n, \ldots$$

Since $R$ can be arbitrarily large we obtain

$$a_n = 0 \quad \text{for} \quad 1, 2, \ldots, n, \ldots$$

implying that the bounded entire function is constant

$$f(z) = a_0.$$ (Liouville theorem).

Using Liouville’s theorem show that:

If $f(z)$ is a polynomial of degree $n \ (n \geq 1)$, then the equation $f(z) = 0$ has a root.

State the residue theorem.

Using the residue theorem show that

$$\int_{-\infty}^{+\infty} \frac{1}{1 + x^2} \, dx = \pi.$$

Would you remember an example for the Weierstrass Factorization Theorem?

Have a good time.
1) Assume that $f$ is a $C^{n+1}$ function on $[a, b]$. Find an error formula in the Lagrange polynomial interpolation, i.e., estimate $f(x) - p_n(x)$ on $[a,b]$. 

2) Formulate the "best approximation" problem for the function space $C[a,b]$ and the subspace $P_n$, the space of polynomials of degree less than or equal to $n$. Prove or disprove existence and uniqueness of solutions of this problem.

3) Apply the Banach fixed-point theorem to show that if $A$ is a diagonally dominant matrix, then both the Jacobi-method and the Gauss-Seidel-method converge.

4) Explore sufficient conditions for the convergence of the Newton method for the solution of systems of nonlinear equations.

5) Give a convergence analysis of the Crank-Nicolson scheme when applied to the heat equation.
Instructions. Please solve the following problems (show all your work). You cannot use textbooks or your class notes. If you have any questions regarding the problems please let me know.

1. State the Van Kampen’s Theorem and use it to find the fundamental group of the two dimensional torus, $\pi_1(T^2)$, and to find the presentation of the fundamental group a closed orientable surface of genus two, $\pi_1(F_2)$, (See Figure 1).

2. Find the presentation of $\pi_1(\mathbb{R}^3 - L)$, where $L$ is the trefoil knot, $3_1$ (See Figure 2), and show that $\pi_1(\mathbb{R}^3 - L)$ is non-abelian.

3. State the definition of a deformation retraction and prove that $S^{n-1}$ is a deformation retraction of $\mathbb{R}^3 - \{0\}$.

4. Find the second homology group, $H_2^\mathbb{Z}(X)$, of the real projective plane $X = \mathbb{R}P^2$.

![Figure 1](image1.png)

![Figure 2](image2.png)