Real Analysis
Qualifying Exam, Spring 2015
Choose 5 questions and indicate your choice (by putting a check mark √) in the table below.

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Problem 1. Denote by $B$ the closed unit ball in the Euclidean space $\mathbb{R}^n$. Put

$\mathcal{A} := \{\varphi \in C(B, \mathbb{R}) : \varphi \text{ is continuously differentiable, } \max\{\|\nabla \varphi(x)\| : x \in B\} \leq 1, \varphi|_{\partial B} \equiv 0\}$

Show that $\mathcal{A}$ is relatively compact in $C(B, \mathbb{R})$. 
Problem 2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $g : [a, b] \to \mathbb{R}$ of bounded variation. Show that for every $x \in [a, b]$, the integral

$$F(x) := \int_{a}^{x} f(t)dg(t)$$

defines a function $F : [a, b] \to \mathbb{R}$ which is continuous at each point of continuity of the function $g$. 
Problem 3. Let \( f_n : [a, b] \to \mathbb{R} \) be a sequence of increasing (i.e. non-decreasing) functions on \([a, b]\) and let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Assume that \( Z \subset [a, b] \) is a subset such that

(i) \( a, b \in Z \),

(ii) \( \bar{Z} = [a, b] \), i.e. \( Z \) is dense in \([a, b]\),

(iii) For every \( x \in Z \) we have \( \lim_{n \to \infty} f_n(x) = f(x) \).

Prove that \( f_n \) converges uniformly to \( f \) on \([a, b]\).
Problem 4. Let $f : [0, 1] \to \mathbb{R}$ be a function differentiable at any point. Is $f'$ a measurable function? Justify your answer.
Problem 5.

(i) Show that

\[ \int_{[0,1]} \left( \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx = \frac{\pi}{4} \quad \text{and} \quad \int_{[0,1]} \left( \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) \, dy = -\frac{\pi}{4}. \]

(ii) Does this contradict Fubini’s theorem? Justify your answer.
Problem 6. Let $I := [0, 1]$. Does there exist an open subset $B \subset I$ which is everywhere dense in $I$ and such that $I \setminus B$ has positive Lebesgue measure?
Problem 7. Consider the sequence

\[ f_n(x) = \frac{n \sin(x)}{1 + n^2 \sin^2(x)} \quad (x \in [0, \pi]). \]

(i) show that \( \{f_n\} \) converges a.e.;

(ii) show that \( \{f_n\} \) does not converge uniformly;

(iii) describe explicitly the set \( A \subset [0, \pi] \) such that \( \mu(A) < \frac{1}{2015} \) and \( \{f_n\} \) converges uniformly on \([0, \pi] \setminus A\) (here "\( \mu(\cdot) \)" stands for the Lebesgue measure on \([0, \pi]\)).
Problem 8. Let $f : [a, b] \to \mathbb{R}$ be a strictly increasing function and put $D := f([a, b])$. Show that $f^{-1} : D \to \mathbb{R}$ is continuous.
Functional Analysis I
Qualifying Exam, Spring 2015
Choose 5 questions and indicate your choice (by putting a check mark \( \sqrt{\) in the table below.

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Problem 1. Let $\mathbb{E}$ be a normed space and $L \subset \mathbb{E}$ be a linear subspace. We put
\[ L^\perp := \{ f \in \mathbb{E}^* : \forall x \in L \quad \langle f, x \rangle = 0 \}. \]
Show that
\[ \text{dist}(f, L^\perp) := \sup \{ \langle f, x \rangle : \|x\| \leq 1 \text{ and } x \in L \}. \]
Problem 2. Let $E$ be a Banach space and $f \in E^\ast$. Show that

$$\text{dist}(x, \text{Ker}(f)) = \frac{|f(x)|}{\|f\|}.$$
Problem 3. Let $\mathcal{H}$ be a Hilbert space.

(a) Show that for two vectors $x, y \in \mathcal{H}$ we have the following:

$$x \perp y \iff \forall t \in \mathbb{R} \quad ||x + ty|| \geq ||x||.$$ 

(b) Assume that $P \in L(\mathcal{H}, \mathcal{H})$ is a projection, i.e. $P^2 = P$. Show that $P$

$$\forall x \in \mathcal{H} \quad Px \perp x - Px \iff ||P|| = 1,$$

i.e. $P$ is an orthogonal projection if and only if $||P|| = 1$. 
Problem 4. Let $\mathbb{E}$ be a normed space and $L \subset \mathbb{E}$ a subspace of co-dimension 1.

(a) Show that there exists $f \in \mathbb{E}^*$ such that $\text{Ker}(f) = L$.

(b) Find a projection $P \in L(\mathbb{E}, \mathbb{E})$ onto $L$, i.e. $P^2 = P$ and $\text{Im}(P) = L$. 
Problem 5. Does there exist a Banach space $E$ such that the space $E^*$ (its conjugate) admits an infinite \textit{countable} basis? Justify your answer.
Problem 6. Let \( M \subseteq l_2 \) be the set of all sequences \( x = (x_1, \ldots, x_n, \ldots) \) satisfying the condition:

\[
\sum_{n=1}^{\infty} n^2 x_n^2 < 1.
\]

Show that \( M \) is a convex set, but not a convex body. (Recall, in a normed space a convex set is a convex body if it admits at least one interior point).
Problem 7. Let $B$ be the space of all continuous functions on the segment $[0, 1]$ equipped with the norm
\[ \|f\| := \left( \int_0^1 |f(x)|^4 dx \right)^{\frac{1}{4}}. \]

(i) Is $B$ separable?
(ii) Is $B$ complete?
(iii) Does there exist a normed space $\mathbb{E}$ such that $\mathbb{E}^* = B$?

Justify your answers.
Name: 

Complex Analysis Qualifying Exam
Spring 2015
Vladimir Dragović and Tobias Hagge

April 17, 2015

**Definition 1** The cross ratio of four complex numbers is defined as:

\[(z_1, z_2, z_3, z_4) := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.\]

**Definition 2** A homeomorphism between two metric spaces is a continuous, injective, and surjective map with the inverse which is a continuous map.

1. Find if it exists a conformal bijection between the upper half plane and each of the following regions:
   (a) The Unit disk \(D = \{x \in \mathbb{C} \mid |x| < 1\}\),
   (b) \(D - D_2\), where \(D_2 = \{x \in \mathbb{C} \mid |x - \frac{1}{2}| \leq \frac{1}{2}\}\).

2. (a) Call all lines and circles of the plane the extended circles. Prove that a fractional linear transformation maps extended circles to extended circles.
   (b) If \(f\) is a fractional linear transformation, prove that it preserves the cross-ratio: \((f(z_1), f(z_2), f(z_3), f(z_4)) = (z_1, z_2, z_3, z_4)\).

3. Let \(f\) be an entire function.
   (a) Show that if \(f\) has a pole or a removable singularity at infinity, \(f\) is a polynomial.
   (b) Show that if \(f\) is a homeomorphism, then \(f(z) = az + b\) for some \(a \in \mathbb{C}^\ast\), \(b \in \mathbb{C}\).

4. If \(D\) is the open unit disk centered at the origin, \(\Omega \subset \mathbb{C}\) a connected open set, show that for any two holomorphic homeomorphisms \(f : D \to \Omega\) and \(g : D \to \Omega\) such that \(f(0) = g(0)\), there exists a constant \(k\) such that for all \(z \in D\), \(f(z) = g(kz)\). What can you say about \(k\)?

5. Let \(p(z) = \sum_{i=0}^{n} a_i z^i\) be a polynomial with complex coefficients, where each \(|a_i| = 1\). Show that all zeros of \(p\) are contained in the closed disk \(D_2\) of radius 2 and centered at the origin.

6. Calculate the integral, carefully justifying each step:

\[\int_0^\infty \frac{dx}{1 + x^r}.
\]

(Hint: One may apply the residue calculus to the integrals along the curves \(\gamma_R = I_R + C_R - I_{w^2R}\), where \(w\) is the root of the equation \(z^r = -1\), with the smallest positive argument; \(I_R\) is the segment from the origin to the point \((R, 0)\); \(I_{w^2R}\) is the segment from the origin to the point \(w^2R\). \(C_R\) is a part of the circle of radius \(R\) centered at the origin, connecting the points \((R, 0)\) and \(w^2R\).)
Abstract Algebra Qualifying Exam
Spring 2015
Tobias Hagge

April 15, 2015

Show all work.

1. Classify all abelian groups of order 60 up to isomorphism.

2. Let $|G|$ be a group of order 30.
   (a) Show that $G$ is not simple.
   (b) Show that $G$ has a cyclic subgroup of order 15.

3. Let $G$ be a finite group, $H \leq G$ such that $[G : H]$ is the least prime which divides $|G|$. Show that $H \trianglelefteq G$.
   Suggested outline: let $X$ the set of subgroups of $G$ which are conjugate to $H$ in $G$. Argue that if $|X| > 1$, $G$ acts trivially on $X$, and obtain a contradiction.

4. (a) Show that if $R$ is a field, the polynomial ring $R[x]$ is a Euclidean domain.
   (b) Show that $\mathbb{Z}[x]$ is not a Euclidean domain.

5. Show that in a principal ideal domain, a prime ideal is maximal.

6. Let $R$ be a ring with 1, $M$ an $R$-module, $N$ a submodule of $M$. Prove that if $N$ and $M/N$ are finitely generated then so is $M$. 
Qualifying Exam: Ordinary Differential Equations I, April 2015
Math 6315

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM
Give clear, rigorous and complete answers with full details in proofs

1. Consider the system

\[
\begin{align*}
    x_1' &= a(t)x_2 \\
    x_2' &= b(t)x_1
\end{align*}
\]

for \( t \geq 0 \). Assume \( a \) & \( b: \mathbb{R}_+ \to \mathbb{R} \) are continuous functions with \( \lim_{t \to +\infty} a(t) = 1 \) and \( b(t) \) is absolutely integrable on \( \mathbb{R}_+ \). Prove that:
(a.) A unique solution to the above system of ODE with initial values \( x_1(\tau) = \xi_1 \) and \( x_2(\tau) = \xi_2 \) exist for all positive and finite values of \( t \).
(b.) If a solution to above system of ODE is such that \( x_1(t) \) is bounded on \( \mathbb{R}_+ \), then

\[
\lim_{t \to +\infty} x_2(t) = 0
\]

(c.) Make use of the above, to show that the given system of ODE has at least one solution which is unbounded on \( \mathbb{R}_+ \).

2. Find the solution to the following initial value problem, by first finding its state transition matrix.

\[
\begin{align*}
    x_1' + 2x_1 + x_2 &= \sin t \\
    x_2' - 4x_1 - 2x_2 &= \cos t
\end{align*}
\]

3. Consider the BVP

\[
y'' + \lambda y = 0, \quad y(0) = y'(0) = y''(1) = 0, \quad 0 \leq t \leq 1
\]

where \( y' = dy/dt \).
(a.) Is \( \lambda = 0 \) an eigenvalue of this BVP?
(b.) Compute the Green's function, when \( \lambda = 0 \).
(c.) Is this problem self-adjoint?
Numerical Analysis Qualifying Exam

Do any 4 out of 6 problems

1) Suppose that $f$ is a function on $[0, 3]$ for which one knows that
   \[ f(0) = 1, \quad f(1) = 2, \quad f'(1) = -1, \quad f(3) = f'(3) = 0. \]
   Estimate $f(2)$, using Hermite interpolation.

2) Determine the quadrature formula of the type
   \[ \int_{-1}^{1} f(t) dt = a_{-1} \int_{-1}^{1/2} f(t) dt + a_0 f(0) + a_1 \int_{1/2}^{1} f(t) dt + E(f) \]
   having maximum degree of exactness $d$. What is the value of $d$?

3) Consider the fixed point iteration
   \[ x_{n+1} = \phi(x_n), \quad n = 0, 1, 2, ..., \] where $\phi(x) = Ax + Bx^2 + Cx^3$.
   Given a positive number $\alpha$, determine the constants $A, B, C$ such that the iteration converges locally to $\frac{1}{\alpha}$ with order $p = 3$.

4) Let $g(x, y) = (f_x + f_y f)(x, y)$. Show that the one-step method defined by
   the increment function
   \[ \Phi(x, y; h) = f(x, y) + \frac{1}{2} h g(x + \frac{1}{3} h, y + \frac{1}{3} h f(x, y)) \]
   has order $p = 3$. Express the principal error function in terms of $g$ and its derivatives.

5) Describe how Newton's method is applied to solve the system of nonlinear equations
   \[ u_{n+k} = h \beta_k f(x_{n+k}, u_{n+k}) + g_n, \quad g_n = h \sum_{s=0}^{k-1} \beta_s f_{n+s} - \sum_{s=0}^{k-1} \alpha_s u_{n+s} \]
   for the next approximation, $u_{n+k}$.

6) Show that the boundary value problem
   \[ y'' + e^{-y} = 0, \quad 0 \leq x \leq 1, \quad y(0) = y(1) = 0 \]
   has a unique solution that is nonnegative on $[0, 1]$. Set up a finite difference method for solving the problem numerically. Discuss the convergence of your scheme.
Qualifying Exam: Choice- Ordinary Differential Equations II  
April 2015, Math 6316

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM  
Give clear, rigorous and complete answers with full details in proofs

1. (a) Determine all the equilibrium points of

\[ \begin{align*}
x_1' &= x_2 + x_1 x_2 \\
x_2' &= -x_1 + 2x_2.
\end{align*} \]

(b) Determine the stability of \( x = 0 \) for the nonlinear system given in part (a) by finding an appropriate Lyapunov function.

2. (a) Analyze the stability properties of the equilibrium point \( x = 0 \) for the following equation.

\[ x' = \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix} x + \begin{pmatrix} (e^{x_1} - 1) \sin(x_2 t) \\ e^{-t} x_1 x_2 \end{pmatrix} \]

(b) For the above nonlinear system, find the tangent spaces to its stable and unstable manifolds at the origin (if they exist).

3. Does the following 2 dimensional system

\[ \begin{align*}
x_1' &= -x_2 + x_1(1 - x_1^2 - x_2^2) \\
x_2' &= x_1 + x_2(1 - x_1^2 - x_2^2)
\end{align*} \]

have a periodic solution? If so, is it orbitally stable? Hint: It may be helpful to look at the problem in polar coordinates.
1. Given a continuous function $f : X \to Y$, $X$ is compact.
   a. Prove that $f(X)$ is compact.
   b. If $Y$ is Hausdorff and $f$ is a bijection, prove that $f$ is homeomorphism.

2. a. Let $f, g : X \to Y$ be two continuous maps and assume that $Y$ is Hausdorff. Prove that the set $\{x | f(x) = g(x)\}$ is closed in $X$.
   b. Prove that $Y$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) | x \in Y\} \subset Y \times Y$ is closed.

3. Prove that the circle $S^1 = \{(x, y) | x^2 + y^2 = 1\}$ is a smooth compact manifold. Does it have a structure of Lie group? Is it connected?

4. Calculate the induced metric (the first fundamental form) on the surface in $\mathbb{R}^3$
   
   $$r(u, v) = (a \cos u \cos v, a \sin u \cos v, a \sin v).$$

   What kind of surface is it? What is the Gauss curvature at the point $u = \pi/3, v = \pi/4$?

5. Given a metric space $(X, d)$. A map $f : X \to X$ is isometry if for all $x, y \in X$ the condition is satisfied
   
   $$d(f(x), f(y)) = d(x, y).$$

   If $X$ is compact prove that every isometry $f$ is bijective and a homeomorphism. Show that if $X$ is not compact, than an isometry is not always a homeomorphism.
Differential Geometry (6309), Qualifying Exam

Friday, April 17th, 2015

1. Calculate the induced metric (the first fundamental form) on the surface in $R^3$
   \[ r(u, v) = (a \cos u \cos v, a \sin u \cos v, a \sin v). \]
   What kind of surface it is? What is the Gauss curvature at the point
   $u = \pi/3, v = \pi/4$?

2. Compute the Christoffel symbols of the surface of revolution
   \[ r(u, v) = (g(u) \cos v, g(u) \sin v, h(u)), \quad (g'(u))^2 + (h'(u))^2 = 1. \]

3. Prove that the circle $S^1 = \{(x, y)|x^2 + y^2 = 1\}$ is a smooth compact
   manifold. Does it have a structure of Lie group? Is it connected?

4. Find the critical points and critical values of the maps:
   (a) \[ f : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (x + y^2, y + z^2); \]
   (b) \[ f : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (xy, z). \]
   (c) Given two vector fields $v_1 = x \partial_x + y \partial_y$ and $v_2 = (x - y) \partial_x + x \partial_y$.
   Calculate the commutator of $v_1$ and $v_2$.

5. Given a metric space $(X, d)$. A map $f : X \to X$ is isometry if for all
   $x, y \in X$ the condition is satisfied
   \[ d(f(x), f(y)) = d(x, y). \]
   If $X$ is compact prove that every isometry $f$ is bijective and a homeomorphism.
   Show that if $X$ is not compact, than an isometry is not always a homeomorphism.
MATH 6320 - QUALIFYING "CHOICE" EXAMINATION
For: Ms. Het Mankad
April 17th 2015
Spring 2015
V. Ramakrishna

• You must show all work to receive full credit. Each Q is worth 20 points.

• No calculators or electronic devices (including phones) allowed.

• Use only pens. No pencils allowed.

• Write your answers only in the space provided, i.e., not on margins, not in the space separating two parts of a question. Further, CLEARLY separate answers to each subpart. Material written at points where they do not belong, or as an afterthought, will be ignored.

• Q 1 Show that $A_{MPI}$ is the matrix $X$ which minimizes $\| I - AX \|_F$.

Q2 State four conditions equivalent for a matrix to be a RPN matrix.

Q3 Suppose $A_{m \times n}$ has rank $r$. Suppose further that the first $r$ left singular vectors, $u_i$, of $A$ have been found. State and justify a formula for the first $r$ right singular vectors, $v_i$, of $A$, which does not require solving an eigenvalue problem.

Q4 Let $A$ be a square matrix of rank $r$. Show that there is a polynomial of degree $r + 1$ which annihilates $A$.

Q5 Let $C$ be the commutator of two square matrices $A$ and $B$. Show that if $C$ commutes with either matrix then $C$ is nilpotent. Use this to show that a matrix $A$ is normal iff it commutes with $C = [A, A^*]$. 