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beyond the imagination

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Connexion...

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This magazine is not a technical journal for mathematicians and statisticians to present the results of their research. Instead, the magazine is intended for a general audience, with the main goal of opening a cultural landscape of mathematics from a perspective different than one presented to students in the classroom. It is the hope of those involved with its publication that readers will gain a better appreciation for the *Connexion* mathematics has to our lives and to our world.

> *Mathematics, rightly viewed, possesses not only truth, but supreme beauty* – **Bertrand Russell**

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Michael Baron, UTD

Cover Page: A recent retiree who understands the importance of education and the central role of Mathematics for technological leadership has appeared at UTD. He wishes to refresh his knowledge of Math so that he will be able to help any future grandchildren with their homework. He is working hard to get the College Algebra and Precalculus problems answered correctly. Can you help him to get the right answers to some of the problems? Send me your hints: jbkehoe@sbcglobal.net.

The cover page was designed by Czech artist Gabriela Novakova.

Grigori Perelman and the Saga of Poincaré Conjecture

Michael Baron and Jozef Przytycki

The beginning of the21st century saw a major breakthrough in Mathematics. After eight years of hard theoretical work, Russian mathematician Grigori Perelman proved the famous *Poincaré conjecture*, thus solving a 100-year old problem that occupied great mathematical minds of the time and was named one of the seven Millennium Prize problems. The problem was deemed so difficult and important that a $1,000,000 prize was designated by Clay Mathematics Institute for its correct solution.

... Grisha Perelman grew up in St. Petersburg, the "Russian North capital", a city with deep and strong mathematical traditions. Leonard Euler, Christian Goldbach, Daniel and Nicolas Bernoulli worked in St. Petersburg in the 18^{th} century, invited by the Russian Academy of Sciences, which was inspired and advised by Gottfried Wilhelm Leibniz. Pafnuty Chebyshev, Sofia Kovalevskaya, and even young Georg Cantor lived and worked in the city in the 19th century. They were followed by Andrey Markov, Alexander Lyapunov, Lev Landau, Leonid Kantorovich, and many others.

Great mathematicians are not born overnight: they grow. What was the mathematics world of St. Petersburg like at the time when Grisha Perelman made his first steps in it? Yuri Matiyasevich had just completed the solution of the 10-th Hilbert problem[1] [4]. Nikolay Nikolskii and his group were carefully verifying and simplifying Louis de Branges' proof of the famous Bieberbach Conjecture [1]. Renowned mathematicians were working at St. Petersburg State University, Steklov Mathematical Institute, as well as Herzen Mathematical institute (where Isidor Natanson and other Jewish mathematicians landed after being denied a position at the University).

Hundreds of St. Petersburg children were going crazy over mathematics, reading mathematical books and solving problems during their after-school free time, time they might otherwise have used for soccer, skiing, movies, and parties. In different parts of the big city mathematical schools, clubs, and circles, were waiting and ready to accept talented and enthusiastic girls and boys, as well as those who were just-interested, into the attractive world of Mathematics. Grigori was not yet ten years old when he successfully participated in a local mathematical competition. This caused his mother, Lubov, a teacher of mathematics at a commerce school, to recommend Grigori to mathematical circles.

Math Summer Camp of August 1981. G. Perelman is second from the right in the front row. Behind him to the left is S. Rukshin. Perelman's students, A. Bogomolnaia, T. Oikhberg, and A. Teplyaev are in the front row, from the left.

[1] Matiyasevich actually proved that the famous Tenth Problem is unsolvable.

Misha B. first met Grisha Perelman at the mathematical circle that each attended twice a week in the Anichkov Palace, built originally for Russian Empress Elizabeth. It was autumn, 1980. Their early acquaintance was rather remarkable.

... We usually came to the mathematical circle to learn some theory or to solve tricky problems, but once in a while we had *mathematical battles*, says Misha. Two teams had to challenge each other on mathematical problems and vigorously debate their solutions. Once we were set for the next math fight, when we discovered that the number of participants was odd, so we could not form two equal teams. We looked around, and there was Grisha Perelman, quietly sitting in the corner immersed, as usual, in his difficult math problems. "Let us invite Grisha to one of the teams", somebody proposed. "How about Grisha *being one of the teams,*" replied our teacher S. E. Rukshin, at once.

Sergey Rukshin, our teacher and leader, devoted his whole career to the creation of math circles and the Mathematical Olympic Movement. His charm and enthusiasm attracted all of us to Anichkov Palace. Every year he led the St. Petersburg team to the National Mathematical Olympiad, where his students always won high awards.

Very soon we realized that no matter whether Grisha joined one of the teams or became one of the teams, when it came to mathematical battle against Grisha, we were doomed.

Grisha Perelman, who was three years our senior, also taught us from time to time. His first group of students included future Russian Olympiad winners Fedor N. and Anna B.

A typical lesson with Grisha was rather peculiar. Grisha sat and solved *his* problems while we were hopelessly trying to solve *our* problems, generously given to us by Grisha. We called these problems "coffins", because only once in a while was anyone able to advance in any one of them. We even tried to collaborate on solving these problems but sooner or later, our entire group gave up on all Grisha's coffins. But trying to squeeze new problems out of Perelman was also to no avail. Certainly, Grisha knew of many more problems that would keep us entertained. However, as we realize now, he strongly believed that mathematicians should not give up on any problem but should try to find its solution by any and all means.

Our mathematical passion continued in the summertime. Every year, most kids from our mathematical circles went to a math summer camp to dive into Mathematics for the whole month. There, some two hours from the city, in cute barracks surrounded by the pine forest, we had mathematics classes from breakfast until lunch and then again from lunch until dinner. Only for the last couple of hours before bedtime did we have our choice between table tennis, books, classical music...and more math. Actually, very few chose math in the evening. Most, like me, chose table tennis. Do you think Grisha went for math?

Grisha Perelman (4th from the left) and his teacher S. E. Rukshin (3rd) near Anichkov Palace.

No, apparently he was fond of classical music concerts that consisted of recordings suggested and supplied by S. E. Rukshin. Later on, I found out that Grisha played the violin, and played it rather well.

At the camp, Grisha taught our group a course. The course seemed rather interesting, and occasionally a few of us even had the illusion of understanding something. The subject was abstract algebra...I guess. At the end, there was an oral exam. I got a D–that was the average grade. Timur O., for example, got an F whereas Sasha T. overscored us with a proud C.

After 8th grade, almost everyone in our group left their schools and landed together in High School #239. Three years before, the group which included Grisha Perelman did the same. This school was very special. Easily the best school of the city, it specialized officially in mathematical subjects. However, all the main disciplines were taught at a very high level. Gifted children from all over the city tried to win the stiff competition and enter the school for two years. For some of them, attending this school meant traveling more than an hour each way every day, six days a week.

When we were students of HS 239, the school was preparing to celebrate its 250-th jubilee! The school alumni included known scientists, politicians, musicians, and actors. Almost every year, 239-ers were among the winners of the International Mathematical Olympiad. Their portraits are in the school's Hall of Fame, and one of them, of course, is Grisha Perelman.

In 1981 Perelman took a second prize in the All Soviet Mathematical Olympiad. He was unhappy with the result and worked very hard not to be second the following year. In 1982 he won the entire Soviet Olympiad, earning a perfect score and the gold medal at the International Mathematical Olympiad in Budapest.

Grisha studied at HS 239 from 1980 until 1982. The 10th grade was the highest grade in Russia at that time, after which 239-ers went straight to the entrance examinations to the nation's best universities.

The renowned University of St. Petersburg was the first choice for many. The university is famous for having eight Nobel Prize winners among its alumni, along with the discoverer of the Periodic Table of Elements, Dmitri Mendeleev, the builder of the first radio receiver, Alexander Popov, and holders of many other scientific achievements.

As the absolute winner of the International Mathematical Olympiad, Grisha Perelman was admitted without any exams. In fact, the Olympiad was held concurrently with the entrance exams, and its participants were always admitted without contest. Until the beginning of Russian Perestroika in 1985, this was practically the only way a Jewish youngster like Grisha could get admission to the leading university.

In the late 1980s, Perelman went on to earn a Candidate of Science degree (the Soviet equivalent to the Ph.D.). His dissertation was titled "Saddle surfaces in Euclidean spaces."

After graduation, he worked at the Steklov Institute of Mathematics of the USSR Academy of Sciences, where his advisors were Aleksandr Aleksandrov and Yuri Burago. Burago recommended Perelman to Mikhail Gromov, and Grigori was invited to spend a few months in the institute IHES in the suburbs of Paris, after defending his dissertation in 1990. In 1991, Gromov arranged an invitation for Grigori to attend a Geometry festival at Duke University, in the United States. After his talk at the festival, Grigori gave presentations at several American Universities, including one at the University of Pennsylvania.

In 1992, Perelman was invited to spend a semester each at the Courant Institute in New York University and the State University of New York at Stony Brook. From there, he accepted a two-year Miller Research Fellowship at the University of California, Berkeley, in 1993. Although Perelman was offered jobs at several top universities in the US, including Princeton and Stanford, he rejected them all and returned to the Steklov Institute in the summer of 1995. At SUNY Stony Brook, Perelman met young Chinese mathematician Gang Tian; they traveled together to seminars at Princeton (IAS). While there, Grigori met American mathematician Richard Hamilton for the first time.

In the 1990s, Perelman was best known for his work in comparison theorems in Riemannian geometry. Among his most notable achievements was a short and elegant proof of the 20-year-old Soul Conjecture of Cheeger and Gromoll (due to this, he was invited to give a talk at the International Congress of Mathematicians in Zurich in 1994.)

In 1996, when Grigori was back in Leningrad, he was awarded the prestigious prize of the European Mathematical Society. He refused to accept the prize, saying that the work he was being honored for was not only not yet finished, adding that the committee was unable to evaluate his work. It is possible that Grigori was already seriously working on the Poincare and Geometrization Conjectures and needed to avoid all distractions.

The next six years of Grigori's life contain two facts worth mentioning: mathematical and personal. In February 2000, Mike Anderson, a mathematician from SUNY Stony Brook who knew Perelman well and who worked on the Geometrization Conjecture, got an e-mail from Perelman asking about one of his papers, or rather about a mistake in it. The short exchange of e-mails stopped when Grigori wrote that he could not open a file send by Anderson. He said that his sister Elena used to help him with this but now she was studying for her Ph.D. in Mathematics in Israel, and that he would not trust his colleagues from Steklov to open files for him.

Around the same time, Perelman left Burago's group over a dispute revolving around a paper Grigori wrote with his younger colleague, A. Petrunin. Grigori, always a perfectionist, was so unhappy with some of references in their joint paper that, the story goes, he was against hiring Petrunin and asked Burago to follow suit. When Burago refused, Perelman left his laboratory, moving to that of senior mathematician Olga Ladyzhenskaya.

In November 2002, Perelman posted the first of a series of e-prints to the arXiv[2]. In this series, Perelman claimed to have outlined a proof of the Geometrization Conjectures, of which the Poincaré Conjecture is a particular case. Perelman's papers instantly attracted great attention from the mathematical community. In April 2003, he accepted an invitation to visit the Massachusetts Institute of Technology, Princeton University, State University of New York at Stony Brook, Columbia University, and New York University, where he gave a series of talks on his work.

In May 2006, a committee of nine mathematicians voted to award Perelman a Fields Medal for his work on Poincaré Conjecture. The Fields Medal is the highest award in mathematics; two to four medals are awarded every four years.

On August 22, 2006, Perelman was publicly offered the medal at the International Congress of Mathematicians in Madrid, "for his contributions to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow." Perelman did not attend the ceremony and declined to accept the medal, making him the first person in history to decline this prestigious prize.

On March 18, 2010, eight years after the proof of the Poincare Conjecture appeared on ArXiv, CMI[3] announced that Dr. Grigori Perelman of St. Petersburg, Russia, is the recipient of the Millennium Prize for his resolution of Poincaré Conjecture. De-

[2], an archive for electronic preprints of scientific papers in the fields of mathematics, physics, computer science, quantitative biology and statistics, which can be accessed via the World Wide Web. In many fields of mathematics and physics (including topology), almost all scientific papers are placed on the arXiv.

[3]Poincarè Conjecture is one of the seven Millennium Prize Problems established by the Clay Mathematics Institute (CMI) in 2000. CMI was founded in September, 1998, by Mr. Landon T. Clay, a Boston businessman, and his wife, Lavinia D. Clay. CMI is dedicated to increasing and disseminating mathematical knowledge.

spite media speculation, there was no official response from Dr. Perelman on the acceptance of this $1 mln. prize. Instead, he announced that the Clay Institute will be the first to know his decision.

Perelman proved the Poincare Conjecture (1904) and its broad generalization, the Geometrization Conjecture of Thurston (1978). Perelman's proof has fundamentally altered two distinct branches of mathematics. First, the proof solved a problem that was causing great discomfort at the core of topology, the mathematical study of abstract shapes, for over a century. The broader result, a proof of the Geometrization Conjecture, is essentially a "periodic table" that brings clarity to the study of three-dimensional manifolds, much as Mendeleev's Table did for chemistry.

Dr. Grigori Perelman in 2000s.

While bringing new results to topology, Perelman's work brought new techniques to geometry. It cemented the central role of geometric evolution equations, powerful machinery for transforming hard-to-work-with spaces into more-manageable ones. Earlier studies of such equations (e.g. by Richard Hamilton) always ran into "singularities" at which point the equations broke down. Perelman dynamited that roadblock.

Henri Poincaré, who posed his problem in 1904, is generally regarded as the founder of algebraic topology. W. Thurston was awarded a Fields medal for his work in 1982 (although the official ceremony was delayed until 1983 because of martial law in Poland; several mathematicians were arrested, interned, or in hiding, including Roman Duda who was also working on the Poincaré Conjecture).

Currently... Grigori Perelman, one of the most famous mathematicians in the world, lives modestly in St. Petersburg, declining multiple interviews and invitations and trying to avoid the storm of intrusive journalists and photographers by any means at his disposal.

Apparently, not everyone wants to be a celebrity. "Money, glory don't interest me. I do have everything I need. I am not a hero of mathematics, I am only a mathematician, and not even a successful one," says Perelman.

Very few people know where Perelman works and what challenges occupy his mind now. Any statements about his private life in a multitude of publications are most likely to be random, unsupported guesses–mere speculation.

School # 239 continues its beautiful scientific and educational traditions and remains a great source of talented students in Mathematics and other fields.

S. E. Rukshin, now in his early 50s, taught in # 239 for a while but ever left his main mission of attracting and training young mathematicians. He still organizes mathematical circles, summer camps, and Olympiads.

Aleksandr Aleksandrov, the most influential advisor of Perelman and once President of St. Petersburg University, passed away in 1999 at the age of 86. Perelman's other advisor, Dmitri Burago, is Professor of Geometry at St. Petersburg University. After Perelman, Burago taught Riemannian Geometry to all the students mentioned in this article. Richard Hamilton is Professor of Mathematics at Columbia University.

Perelman's students... Timur Oikhberg is now an Associate Professor of Mathematics at the University of California, Irvine. Sasha Teplyaev is an Associate Professor of Mathematics at the University of Connecticut, Storrs. Anton Petrunin is an Associate Professor of Mathematics at Pennsylvania State University. Anna Bogomolnaia is an Associate Professor of Economics at Rice University. Fedor Nazarov, a Salem prize winner, is a Professor of Mathematics at the University of Wisconsin, Madison. Misha B., or Michael Baron, is (proudly) a Professor of Statistics at UT-Dallas.

References

[1] Louis de Branges, Underlying concepts in the proof of the Bieberbach conjecture, Proceedings of the International Congress of Mathematicians, Vol. 1, 2, Providence, R.I.: American Mathematical Society, pp. 25-42, 1987

[2] Masha Gessen, Perfect rigor; A genius and the mathematical breakthrough of the century, Houghton Mifflin Harcourt, Boston-New York, 2009.

[3] Graham Farmelo, The Strangest Man: The Hidden Life of Paul Dirac, Mystic of the Atom, Basic Books, 539 pp.

[4] Yu. V. Matiyasevich, Hilbert's Tenth Problem, The MIT Press, Cambridge, 1993.

[5] S. Nasar, D. Gruber, Manifold Destiny: A legendary problem and the battle over who solved it, *New Yorker*, August 28, 2006.

[6] D. O'Shea, *The Poincaré Conjecture: In search of the shape of the universe*, Walker Publishing Company, New York 2007.

[7] G. Perelman, The Entropy Formula for the Ricci Flow and its Geometric Applications; e-print put in arXiv November 11, 2002: arXiv.org: math.DG/0211159 (2002)

[8] G. Perelman, Ricci Flow with Surgery on Three-Manifolds; e-print put in arXiv March 10, 2003: arXiv.org: math.DG/0303109 (2003)

[9] G. Perelman, Finite Extinction Time for the Solutions to the Ricci Flow on Certain Three-Manifolds; e-print put in arXiv July 17, 2003: arXiv.org: math.DG/0307245 (2003)

[10] G. G. Szpiro, Poincareé Prize; the hundred-year quest to solve one of math's greatest puzzlees, A Plume Book, 2007.

Measuring our Universe

Larry P. Ammann

> *"A remarkable relation between the brightness of these variables and the length of their periods will be noticed."*

These simple words, written nearly 100 years ago, heralded a discovery that profoundly changed mankind's view of our Universe – its size, its beginning, its composition, and its future. To understand properly the importance of this discovery and the remarkable person responsible for it, we first must review how distances in our Universe were measured prior to the beginning of the 20^{th} century.

Solar System Distances: Earth and Moon

In the 3^{rd} century BCE, the Greek mathematician Eratosthenes was able to estimate the Earth's circumference. He learned that on the summer solstice the Sun at noon was reflected in a deep well in the southern Egyptian city of Syene, which indicated that the Sun was directly overhead at that time. He also knew the Sun was not directly overhead at that time where he lived (Alexandria, in northern Egypt) but was about 1/50 of a circle (about $7.2°$) below the zenith. Since Alexandria was about 550 miles north of Syene, then Eratosthenes estimated that the Earth's circumference was about $550 * 50 = 27500$ miles, a size that is only about 10% higher than the actual circumference.

A distance that required more mathematically sophisticated methods was derived during the 2^{nd} century BCE by the Greek mathematician Hipparchus. He applied his newly developed mathematics, now called trigonometry, to observations of a Solar Eclipse at two different sites to estimate the Earth-Moon distance in terms of the radius of the Earth. Figure 1 illustrates a simplified version of the geometry of these observations. It assumes that the Eclipse occurred at noon, the two sites were on a north-south line, and the Sun is infinitely distant. Correcting for these details won't have much impact on the final answer.

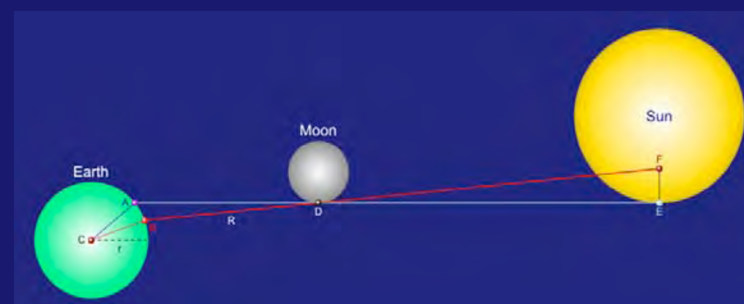


Figure 1: *Geometry of a Solar Eclipse from Two Locations*

At position A (Hellespont, Greece) the Moon completely eclipsed the Sun, but at position B (Alexandria, Egypt), the Moon only partially eclipsed the Sun, leaving approximately 1/5 of the Sun's diameter uncovered. Hipparchus knew the latitudes of these two sites (his values were $41°$ at A and $31°$ at B) and he also knew that the full diameter of the Sun covered approximately 1/650 ($0.554°$) of its circular path around the sky. This implies that $\angle EDF = \angle CDB = 0.111°$. Since AB is the base of an isosceles triangle with vertex C, then $\angle CAB = 85$. At the time of this eclipse, the Sun and Moon were $3°$ below the Celestial Equator, and so the angle between the Moon and the Zenith at A would have been $41 + 3 = 44$. Therefore, $\angle BAD = 180 - 85 - 44 = 51$. We now can apply the law of sines to the oblique triangle ABD,

$$\frac{\sin(DAB)}{R} = \frac{\sin(ADB)}{AB}.$$

Also, AB is a chord of the angle ACB, and so

$$AB = 2r\sin(CAB/2).$$

These results give,

$$\frac{R}{r} = \frac{2\sin(51)\sin(5)}{\sin(.111)} = 69.9.$$

The Earth-Moon distance is usually expressed as the distance between their centers, and so these calculations give this distance as approximately 71 times the radius of the Earth. The actual distance is about 60 Earth radii, so this was a reasonably good approximation given the limitations of the observations.

Note. Hipparchus did not know about sines, but he did know about chords and the law of chords for oblique triangles that is

equivalent to the law of sines. Also, he would have used the approximation,

$$\text{Chord}(\theta) = 2\sin(\theta/2) \approx \theta,$$

when θ is small for the very small angle $\angle ADB$ and with θ expressed in radians.

Note. If we combine the value Eratosthenes obtained for the Earth's circumference and the value Hipparchus used for π,

$$\pi \approx \frac{(60)(180)}{3438} = 3.141361,$$

then the distance of the Moon from Earth's center is

$$R \approx \frac{(70.9)(27500)(3438)}{(60)(360)} = 310,000.$$

The actual mean distance is about 240,000 miles. In addition, we can derive the Moon's diameter from these results. Since the Moon has the same angular size as the Sun, $(0.554°)$, then the diameter of the Moon can be expressed as a chord of this angle. Hence, these results give

$$D_m \approx \frac{(69.9)(.554)(27500)}{360} = 2958,$$

compared to the actual diameter of the Moon which is 2159 miles.

The methods used by Hipparchus were correct. The main source of error in his computations was the imprecise value given to the angle $\angle ADB$. This angle, which represents the angular change in position of an object relative to a very distant background seen from two different vantage points, is called the *parallax* of the object. The method Hipparchus used to derive the moon's distance is called the *Parallax Method*. It is still used today.

Solar System Distances: the Sun

A more difficult problem was to determine the Earth-Sun distance, called the Astronomical Unit (AU). It is possible to derive distances to other planets in terms of the AU using the same methods Hipparchus used, but their actual distances could not be derived until the AU was known. The Parallax Method can't be used directly to obtain AU because the Sun is so bright it was not possible to observe any distant objects, such as other stars, behind the Sun. However, an earlier Greek mathematician, Aristarchus (310-230 BCE) realized that when exactly half of the Moon is illuminated as seen from Earth, then the angle formed by the Earth-Moon-Sun must be a right angle. By measuring the apparent angle between the Moon and the Sun, Moon-Earth-Sun, he could express the distance to the Sun, ES, in terms of the distance to the Moon, EM. Aristarchus measured this angle to be $87°$, and so the Sun's distance relative to the Moon's distance based on this value is,

$$\frac{ES}{EM} = \frac{1}{\cos(87)} = 19.$$

Unfortunately, his value for the Moon-Earth-Sun angle was not very accurate, and so $ES = 19(EM)$ is considerably short of the actual distance, $ES = 400(EM)$. Furthermore, it was difficult for him to determine precisely when the Moon was exactly half-illuminated. A more accurate determination of AU would have to wait until better instruments were developed. Nevertheless, it was a remarkable and audacious achievement of these men to first ask the questions, how distant are the Moon and Sun, and then to apply valid mathematical methods in their attempts to answer those questions.

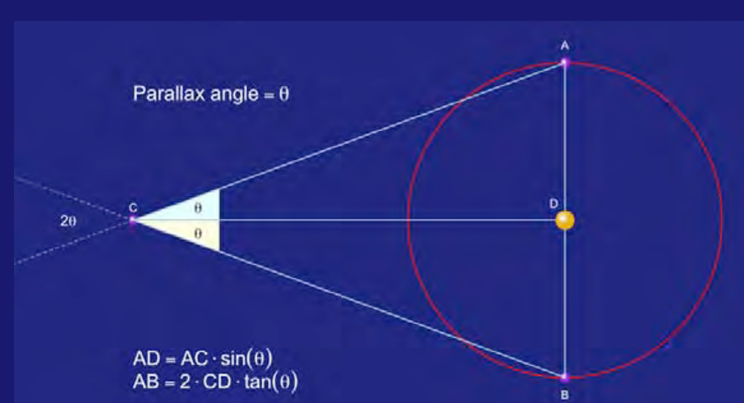


Figure 2: *Parallax Method*

The Parallax Method of distance determination is based on the simple relationship between the angular change in the position of an object relative to a very distant background when viewed from different vantage points and the distance between those vantage points. This is illustrated in Figure 2. An object at **C** is observed at points **A** and **B**. The apparent shift in position of **C** against a distance background is measured by the angle θ. The distance **AD** is called the *baseline*. This method is limited by our ability to measure very small angles, and so to determine the distance to very far objects, we must use as long a baseline as possible. Astronomers realized that the longest possible baseline that could be used to measure astronomical distances is obtained by measuring positions of objects when the Earth is at opposite points in its orbit around the Sun. That is, if the Sun is at **D** and **C** is observed at two times, **A**, **B**, that are six months apart, then the baseline for parallax measurements would be AU. Of course, this requires an accurate determination of AU.

When Galileo constructed his telescope and discovered that Venus has phases like the Moon, and that Jupiter has Moons of its own orbiting it like a miniature solar system, the idea of a universe with our Earth at its center was completely and irrevocably shattered. It also became clear that the telescope could help overcome the problems encountered by Greek astronomers by providing more precise measurements of stellar positions and parallax angles. This new instrument, combined with the mathematical tools of trigonometry and the laws of planetary motion–discovered by Johannes Kepler and eventually justified by Isaac Newton's Theory of Gravity–provided new opportunities to measure our Solar System and beyond.

The Italian-French astronomer Giovanni Domenico Cassini, recognized that an upcoming opposition of Mars in 1672 would provide an opportunity for accurately measuring the parallax of Mars, which in turn could be used to obtain the Sun's parallax. When Mars is at opposition, that is, when it is directly opposite the Sun as seen from Earth, it comes closest to Earth, and so its parallax is larger than at any other time in its orbit. To provide a large baseline from which to obtain this parallax, Cassini sent a young French astronomer, Jean Richer, to Cayenne in South America. Richer was assigned to measure the position of Mars relative to nearby stars from Cayenne at the exact time of opposition while Cassini did the same from Paris. Once Richer returned to Paris, Cassini combined the two data sets and determined that the parallax of Mars was 25 arcseconds. Since the distance between Cayenne and Paris is about 4000 miles, then Cassini computed the distance between Earth and Mars at the time of these measurements to be

$$4000\frac{180*3600}{25*\pi} = 33 \text{ million miles.}$$

Kepler's 1^{st} law of planetary motion states that planetary orbits are ellipses and his 3^{rd} law states that the square of a planet's orbital period is directly proportional to the cube of the semi-major axis of its orbit. That is,

$$\left(\frac{P_m}{P_e}\right)^2 = \left(\frac{R_m}{R_e}\right)^3.$$

However, because Mars has a much greater orbital eccentricity than Earth, the distance between Mars and Earth at opposition can vary significantly, depending on where the two planets are in their orbits relative to their aphelion and perihelion positions. In the case of the 1672 opposition, Mars was close to its perihelion (closest to the Sun) and Earth was close to its aphelion (furthest from the Sun). Cassini determined that Mars was approximately 0.38 AU from Earth at that time. He then derived a value for AU of

$$33/0.38 = 86.8 \text{ million miles,}$$

very close to the accepted mean distance of 93 million miles. That was staggering news at the time, for it expanded the size of our Solar System by twenty-fold over the value obtained by Aristarchus.

Kepler's 3^{rd} law gives us the ability to obtain the distance of any planet to the Sun in terms of Earth's distance, the AU, from that planet's orbital period, its year. By obtaining an accurate value of AU, Cassini unlocked the distances and sizes of other planets in the solar system. This gave us, for the first time in the history of our species, an accurate understanding of the scale of our Solar System. For example, using the modern value for the average angular diameter of the Sun of $0.533°$, we find that the Sun's diameter is 865,000 miles (see Figure 2). Jupiter's orbital period is 11.86 years, and so is $11.86^{2/3} = 5.2$ AU from the Sun. That corresponds to 483 million miles. Knowledge of its distance enables us to derive Jupiter's diameter from its angular size. This gives an equatorial diameter of 88,700 miles, about 11 times the Earth's diameter. Saturn is almost twice Jupiter's distance from the Sun at 886 million miles, with an equatorial diameter of 74,900 miles. The scale of our Solar System is difficult to comprehend, and yet the center of it, our Sun, is just one of what appear to be uncountably many stars in the sky. How far away are they?

Distances to the Stars and Beyond

As telescopes were improved and made larger, and as finely-machined measurement instruments were attached to these telescopes, it became possible to obtain parallaxes of nearby stars using a baseline that stretched across opposite positions in Earth's orbit (see Figure 2). To illustrate the very small angles these instruments were required to measure, imagine drawing a circle, and then partitioning it into 360 equal arcs. The angular size of each arc is 1°. Now partition each of those arcs into 60 equal subarcs. The angular size of these subarcs is 1 arcminute. Further subdivide each of those into 60 equal subarcs and you now have an arc whose angular size is 1 arcsecond. So an arcsecond is 1/3600 of a degree, and there are 1,296,000 arcseconds in a full circle.

Astronomers define a *parsec* to be the distance of an object that has a parallax of 1 arcsecond relative to a baseline width of one AU. Since the sine and tangent of a very small angle are essentially the same as the angle expressed in radians, then 1 parsec corresponds to a distance in miles of

$$\frac{9.3E7 * 3600 * 180}{\pi} = 1.918 * 10^{13}.$$

Light travels 186,000 miles per second in the vacuum of space, and so the time it takes light to travel 1 parsec is $1.03 * 10^8$ seconds, or 3.26 years. That is, one parsec is equivalent to a distance of 3.26 light-years. The challenge to astronomers is that no star has a parallax as large as 1 arcsecond.

It was not until 1837-1839 that telescopes and parallax-measuring instruments became sufficiently powerful and accurate to permit the determination of a star's parallax. Friedrich Struve published his measurement of the parallax of Vega, the brightest star in the constellation *Lyra*, as 0.125 arcseconds (26 light-years) in 1837, but the reliability of his measurement was poor, even though his first published value is very close to the currently accepted value of 0.129 arcseconds. In 1838 Friedrich Wilhelm Bessel obtained a very accurate parallax of 0.314 arcseconds (10.4 light-years) for the star 61 *Cygni*. Shortly thereafter in 1839, Thomas Henderson published his parallax of Alpha Centauri, 0.75 arcseconds (4.3 light-years).

These successful measurements of stellar parallaxes encouraged observatories, with funding from wealthy philanthropists, to build increasingly larger telescopes that could extend our knowledge of the scale of the Universe. Four such large telescopes were built in the United States. The University of California's Lick Observatory added its $36''$ refractor in 1889, and in 1897 the University of Chicago's Yerkes Observatory completed its $40''$ refractor. In 1908 Mt. Wilson Observatory completed its $60''$ reflector, at that time the largest in the world. It later added the $100''$ Hooker reflector in 1917. These classically beautiful telescopes were able to obtain many more stellar parallaxes. Yerkes Observatory also pioneered the use of photography for the determination of stellar parallax. This produced a 10-fold increase in the accuracy of these measurements and significantly extended the size of the known Universe.

The new telescopes that were built in the late 19^{th} and early 20^{th} century made many other significant discoveries. They also contributed to a great mystery that puzzled astronomers of this time.

Figure 3: *M20, The Trifid Nebula in Sagittarius*

Figure 4: *M13, Globular Cluster in Hercules*

On late fall and early winter evenings, high overhead in the constellation *Andromeda*, there is a small fuzzy patch in the sky that can be seen with the unaided eye away from city lights. This object, known as M31 from Charles Messier's catalog, is one of largest examples of what were referred to as *spiral nebulae* by late 19^{th} and early 20^{th} century astronomers due to their noticeable spiral structure. As the new telescopes at Lick, Yerkes, and Mt. Wilson began close scrutiny of Spiral Nebulae, some astronomers started getting hints that these objects were not just clouds of gas, but instead contained stars. Astronomers were familiar with open star clusters such as the Pleiades, globular clusters such as M13 in *Hercules*, and ordinary Gaseous Nebulae such as M20, the Trifid Nebula, in *Sagittarius*. But Spiral Nebulae were very different from other objects and their sizes were unknown because their distances were unknown. What was the nature of these objects and how distant were they?

Figure 5: *M31, Spiral Nebula in Andromeda*

Unfortunately, there is a physical limit in our ability to measure stellar parallaxes. The wave nature of light and the optics of telescopes cause these point sources to appear as disks, called Airy disks, due to diffraction. Earth-bound telescopes are further limited by atmospheric distortions that smear stellar images and cause them to move around under high magnification, much like the shimmering air above an asphalt highway on a hot day. So questions about the nature of spiral nebulae sat unanswered, waiting for a great leap in our ability to measure the universe. That leap would come from an unlikely source.

Henrietta Leavitt and Cepheid Variables

At the beginning of the 20^{th} century, Astronomy, along with the rest of Science, was on the cusp of a major revolution in our understanding of the Universe and how it functions. Earlier in 1814, Joseph von Fraunhofer had closely examined dark lines that appeared in Solar spectra and carefully measured their wavelengths. Then in 1859, Gustav Kirchhoff and Robert Bunsen showed that these dark lines were located at the same wavelengths of light that were emitted by various chemical elements, thus proving that Fraunhofer lines represent absorption of light from the Sun by these elements (see Figure 6). This was the beginning of a new science, called *spectroscopy*, that enabled astronomers to identify the chemical composition of stars without obtaining a physical sample from them. However, the reason atoms only absorbed or emitted specific wavelengths of light was not yet known. James Clerk Maxwell published his equations that characterize electromagnetic fields in a series of papers beginning in 1861. Classical Electromagnetic theory is based on these equations and they imply that light is an electromagnetic wave. But it was difficult for scientists of that time to understand how the energy of an electromagnetic wave could move across the space between stars without something that is waved, like an ocean that carries the energy of ocean waves.

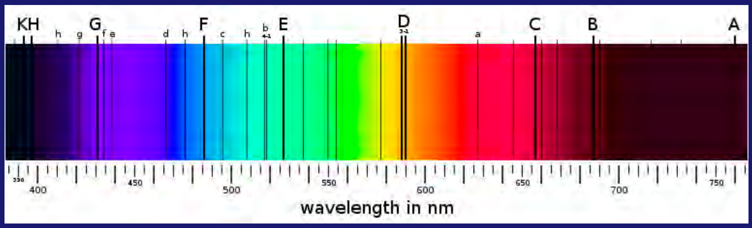


Figure 6: *Light from the Sun is passed through a prism to spread out its colors. Fraunhofer lines are the black lines and represent missing wavelengths of light that have been absorbed by atoms before reaching Earth.*

In 1905, Albert Einstein published three remarkable papers. The first paper, on the photoelectric effect, provided a basis for Quantum theory. The second, his Special Theory of Relativity, describes the nature of space and time and the strange behavior of matter when it travels close to the speed of light. His third paper solved the problem of Brownian motion and proved the existence of atom-sized molecules. In 1913, Niels Bohr introduced his model for atoms which was later refined by Quantum Mechanics and which explained Fraunhofer lines. In 1915, Einstein published his General Theory of Relativity which was verified in spectacular fashion by the 1919 Solar Eclipse. Sir Arthur Eddington's expedition to the island of Príncipe to photograph stars near the Sun during totality proved that their apparent positions had been perturbed by amounts correctly predicted by the General Theory. Einstein's description of the equivalence of mass and energy, expressed by the iconic equation $E = mc^2$, would lead eventually to understanding how stars are fueled.

The Universe, however, was assumed to be static with some individual objects, such as comets, novae and variable stars, changing, but with a basic structure that did not change. Our Universe was referred to as the Milky Way after the faint band of light that is best seen in summer from a dark location. As-

Figure 7: *Large and Small Magellanic Clouds with the Milky Way at Cerro Tololo Observatory in Chile*

tronomers could see that the Milky Way contained a variety of different objects, but most of their distances were unknown. Although spectroscopy could determine the chemical composition of stars and ordinary nebulae, with no means to measure distances beyond a few hundred light-years from Earth, the true nature of our Universe was unknowable and unimaginable.

There is a reason why all of the names mentioned thus far are the names of men. Relatively few women attended college then, and those who did had very limited opportunities to apply their knowledge and skills other than by teaching or nursing. One exception to this was provided by Harvard Observatory. Edward Charles Pickering was appointed director of the observatory in 1877 and served in that position until his death in 1919. As photography began to replace visual observation in the late 19^{th} century, data in the form of photographic plates was accumulating much faster than it could be catalogued and analyzed. Pickering recognized that a hitherto untapped resource consisting of intelligent, well-educated women, could provide valuable assistance in the tedious work of cataloguing this ever-expanding collection of astronomical data, and by performing related computations. Of course, it is likely that the success of his arguments to the Harvard Corporation to hire women was enhanced by the fact that women could be hired at a considerably lower salary than men, and that these women would not balk at being assigned to perform tiresome, uninteresting duties associated with data-reduction and cataloguing.

One of these women was Henrietta Leavitt, born in 1868 and educated at what is now known as Radcliffe College. Her family was sufficiently wealthy to afford her education and Miss Leavitt did not need to work. After graduation, she traveled around the country and Europe, but became ill and eventually lost her hearing. She had fallen in love with Astronomy during her senior year of college and so, after returning home to Cambridge, Massachusetts, she began volunteer work at Harvard Observatory in 1895. Seven years later she was hired by Pickering at a salary of $.30/hour. That salary was less than what Pickering paid his secretary.

Pickering gave Miss Leavitt two principal assignments. The first was to develop a method to derive stellar magnitudes (apparent brightness) from photographic plates, and the second was to identify and catalogue variable stars. To complete the first assignment, she devised and then applied new analytical tools to identify stars near the North Celestial Pole that comprised a sequence of standard magnitudes for comparison with other stars in their vicinity. She then extended this work to identify standard sequences in other sections of the sky. Her standards were utilized until technology advanced sufficiently to permit photometric measurements. This work facilitated her search for variable stars and she discovered over 2400 variables.

Henrietta Leavitt possessed an inquiring intelligence that was not satisfied with simply transferring information from photographic plates to a written catalogue. She also began to see patterns in this data and wanted to understand their meaning better. One particular set of plates she analyzed came from Harvard's southern hemisphere Observing Station and contained photographic surveys taken over 13 years of the Small and Large Magellanic Clouds. The SMC and LMC are two clusters of stars that can be seen only from the southern hemisphere or low northern latitudes. Armed with her newly developed methods for magnitude determination from photographic plates, Leavitt was able to identify in the SMC 25 members of a type of variable star now called a Cepheid variable.

A Cepheid variable changes brightness in a periodic way that can be recognized by its light curve, the relationship between its brightness and time. Since the Cepheids Henrietta Leavitt had discovered were located in the SMC, she correctly reasoned that they were all essentially the same distance from Earth. This implied to her that the differences in brightness among these stars represented differences in their intrinsic brightness, or luminosity. Her curiosity led her to plot the period of these Cepheids versus their brightness. Figure 8 displays her data.

As Miss Leavitt wrote in the 1912 paper that announced her discovery, "A remarkable relation between the brightness of these variables and the length of their periods will be noticed." Why was this relationship important to Astronomy? Imagine having a light bulb and a meter that can measure light intensity. We know from the inverse square law that the energy of light decreases with the square of the distance from the light source. Suppose we use the meter to measure the apparent brightness of this bulb at a distance accurately measured to be

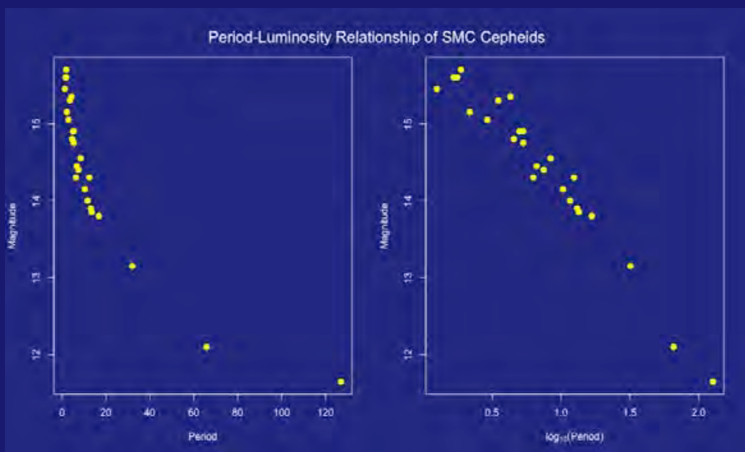


Figure 8: *Henrietta Leavitt's Data*

Figure 9: *M51, the Whirlpool Galaxy*

100 feet and obtain a brightness value of 20. Now suppose we move the bulb to a different location, aim the meter at the bulb, and obtain a reading of 5 for its apparent brightness. Since this value is $1/4$ of the brightness at 100 feet, then the distance must be $100\sqrt{4} = 200$ feet. Comparing the luminosity of a star to its apparent brightness gives distance to the star in a similar way. The relationship between period and luminosity discovered by Henrietta Leavitt made possible the ability to determine luminosity of Cepheids from their periods. A comparison of the luminosity of a Cepheid to its apparent brightness could give its distance. It remained to calibrate the Period-Luminosity relationship of Cepheid variables. As noted by Miss Leavitt in her paper, "It is to be hoped, also, that the parallaxes of some variables of this type may be measured."

Leavitt was not free to pursue her own interests at the Observatory, and so she continued her work to catalogue information recorded by other astronomers. One year after her announcement, Ejnar Hertzsprung obtained the parallaxes of several Cepheid variables in the Milky Way, thus providing an initial calibration of the Period-Luminosity relationship of Cepheid variables. Harlow Shapley extended this work by measuring the parallaxes of a much larger number of Cepheids to recalibrate the P-L relationship. He then used his recalibrated yardstick to measure distances to globular clusters that surround the Milky Way. His work put limits on the size of the Milky Way. After Pickering's death, Shapely was named director of the Harvard Observatory in 1921 and then promoted Miss Leavitt to be head of Stellar Photometry. Later that year she died of cancer.

Sadly, Henrietta Leavitt did not live long enough to witness the greatest triumphs of her discovery. In 1925, Edwin Hubble announced his identification of Cepheid variables in the Andromeda Spiral Nebula using photographs taken with the Hooker telescope at Mt. Wilson. The distance derived by Hubble from the Cepheid P-L relationship proved conclusively that this object is far outside the Milky Way. Astronomers now understood that Spiral Nebulae were in fact spiral *galaxies*, separate and very distant from our own Milky Way Galaxy. Hubble then identified Cepheid variables in other galaxies, thus providing him with their distances. He combined this information with spectroscopic data and noticed that the Fraunhofer lines of galaxies were shifted toward the red by amounts that were directly related to their distances. This galactic *red-shift* provided the first evidence for the expansion of our universe from a *Big Bang* beginning, now known to have occurred about 13.7 billion years ago.

Henrietta Leavitt and the other women who worked at Harvard Observatory made significant contributions to Astronomy in spite of strong cultural prejudices of that time against the idea of women gaining advanced academic degrees and holding professional positions. Their successes helped pave the way for other women to enter previously male-dominated scientific fields and confirmed that society benefits most when it allows full use of all of its collective intelligence. Under the leadership of Harlow Shapely, Harvard Observatory created a fellowship that enabled women to study Astronomy at the Observatory. One of the first recipients of that fellowship later became the first woman to chair an academic department at Harvard. The past 100 years have seen spectacular advances in our understanding of the universe and how it functions, much of which stands on the shoulders of these remarkable women.

Further reading

The 1912 paper announcing Henrietta Leavitt's discovery of the Period-Luminosity relationship of Cepheid variables can be found here.
http://www.physics.ucla.edu/~cwp/articles/leavitt/leavitt.note.html
Note that the author is listed as Edward C. Pickering. Because Miss Leavitt was not a member of the Observatory faculty, she was not permitted to submit papers to the **Circular**. However, Pickering immediately gives credit to Miss Leavitt in the paper.

More information about Henrietta Leavitt and the other women who worked at Harvard Observatory during this time can be found here.

1. Johnson, George. 2006. **Miss Leavitt's Stars: The Untold Story of the Woman Who Discovered How to Measure the Universe**. W.W. Norton, London.
2. http://www.womanastronomer.com/harvard_computers.htm
3. http://outreach.atnf.csiro.au/education/senior/astrophysics/variable_cepheids.html

The following references give more detailed technical information regarding the Period-Luminosity relationship of Cepheid variables and its calibration.

1. Koen, C., Kanbur, S. and Ngeow, C. 2007. *The detailed forms of the LMC Cepheid PL and PLC relations*. Mon. Not. R. Astron. Soc. **380**, 1440-1448.
2. Madore, B.F. and Freedman, W.L. 1998. *Calibration of the Extragalactic Distance Scale*. **Stellar Astrophysics for the Local Group: VIII Canary Islands Winter School of Astrophysics**. Ed. A. Aparicio, A. Herrero, and f. Sanchez. Cambridge University Press.

Last, but not least, you can find new images of our Universe every day at the Astronomy Picture of the day:
http://antwrp.gsfc.nasa.gov/apod/astropix.html
Many of the images included here were displayed at APOD.

Larry Ammann, UTD

Figure 10: *M104, the Sombrero Galaxy*

Figure 11: *M77 and NGC 5394*

Solving Problems Can Be Fun

Ambikeshwar Sharma

Sometimes simple mathematical problems are put to you at a time and at a place when you least expect them. I believe that it pays to face them and to try to understand them and if possible to solve them. As a student of mathematics it is not fair to refuse to attend to the problem or to pass if off with a contemptuous wave of the hand. This conviction came to me by an event that happened to me many years ago when I was going by an evening overnight train from Lucknow to Allahabad, India.

When I came to the University of Edmonton in Alberta, Canada this conviction became all the more strongly entrenched in my mind. In Edmonton, I met Professor Leo Moser (http://en.wikipedia.org/wiki/Leo_Moser) for the first time. Professor Moser was a simple, courteous and soft-spoken person, full of anecdotes, humor and problems. Although the mathematics department was small, lacking its own building, Leo Moser was the centre of activity. He freely discussed problems with anyone who cared to listen.

He had a photographic memory and was an excellent chess player. I learned that he could play chess against 20 or 30 teams of students from different schools at the same time and would win against all of them. In the faculty lounge in the department, he would discuss problems or tell anecdotes to his students. Anyone who came in and wanted to listen was welcome.

One day I heard him talking to a student about Blichfeldts Lemma. I had heard about it but I did not know what it was and why it was important. Professor Moser then explained to me how this lemma was proved by an American mathematician, Blichfeldt. It states that if there is a plane region R of any shape with an area more than n units, then it is always possible to translate (i.e., slide without turning) it in such a way that it covers $n+1$ lattice points (with integer coordinates). In particular, he explained by a sketch on the blackboard that for $n = 1$, there is a pair of distinct points A and B with coordinates (x_1, y_1) and (x_2, y_2) such that $x_2 - x_1$ and $y_2 - y_1$ are both integers. He explained to me another time how this lemma was used by Minkowski to solve a problem of Hilbert about an orchard called the orchard problem (http://mathworld.wolfram.com/Orchard-PlantingProblem.html).

Like the Scottish Problem Book (http://www.gap-system.org/~history/HistTopics/Scottish_Book.html) of problems in Lwów (formerly Poland), he had started a book of problems in which visitors and anyone who has a problem could put his problem in black and white. His contacts with people like Martin Gardner, Professor P. Turán, Professor P. Erdös, Professor D.J. Newman and many others brought us the visits of some of these well known people. Once I showed him a poem called "Song of a Ph.D.," a parody written on the lines of Gilbert and Sullivan, which I had heard at Cornell. He read the poem and could recite it the next day. I still recall the first stanza, which runs like this:

When I was a kid and went to school,
Arithmetic was taught by rote and rule,
I did long division and I did cube roots,
At the Rule of Three, I was specially astute,
I was so astute at the Rule of Three,
That now I am the holder of a Ph.D.

Professor Moser was very hospitable and the evening parties at his home were always a treat. His passing away at an early age due to a heart attack was a serious blow to the department. One of his last students Professor David Klarner is known for his work on Polyominoes (http://en.wikipedia.org/wiki/Polyomino).

To return to the circumstance of the event that happened to me in India, when I was working at the University of Lucknow: I wanted to go to Allahabad by the overnight evening train and to consult the university library there during the day and return the next evening back to Lucknow. I could not afford the luxury of a first class or second class ticket and so bought a third class return ticket. I arrived at the railway station half an hour earlier than the scheduled departure time in order to acquire an upper berth (if possible) and at best a comfortable seat away from the tumult and rush of passengers. I decided to take a seat that looked promising but noticed that a gentleman was ambling outside with an eye on his suitcase. He had already occupied the upper berth and had spread his blanket there for the night and so I had to occupy the lower berth.

Figure 12: *A view of the University of Allahabad*

A few minutes later the gentleman came in and I learned from him that he was a businessman who was going to Allahabad on some business. He had a big store in a fashionable area in Aminabad. I told him that I was a lecturer at the university and taught mathematics. He seemed happy to learn this and asked me if I could solve two questions for him that his son had asked him and that he could not do. I told him that I would give his problems a try and invited him to state them. My companion started telling me the first problem.

PROBLEM 1. A man was badly in need of an honest hard working servant to look after his cows and do all the household work, as his wife was sick and could not manage the job. He was willing to pay him food and lodging and a dollar a day, paid monthly, but the servant must do all the jobs. He confided his problem to a close friend of his who promised to look around and find a good chap. In a few days his friend, a goldsmith by profession, brought him a young sturdy fellow who was willing to do all the work for the salary offered, except that there would be a condition that he wanted the master to accept. The condition required by this servant was that if he decided to leave on a certain day, he must get the exact salary up to that day. If the master were unable to pay the exact amount up to that day, the master would have to pay a severe penalty of losing some body part (nose and ears). Since the man needed a servant badly, he agreed to the terms without much thought. The servant did prove to be excellent and he did all the jobs well without murmur or dissent. But the master began to worry about the terms imposed by the servant and this worry made him sick. He again told his difficulty to his friend, the goldsmith, who asked him to be of good cheer. He asked him to give him $31 and in return he gave him five gold rings, which he was asked to put on his fingers. My companion asked me to tell him the price of each of those golden rings with which he could pay his strange servant if he decided to leave on any day of the next month.

I went over the problem with him again to get some time to think. By this time other passengers were streaming in and our compartment was getting filled up. After a few minutes I was lucky to get the solution for my friend and when I told him the price of each of the five rings, he was happy. His second problem was as follows:

PROBLEM 2. Three men with a monkey bought some mangoes, but decided to eat the mangoes next morning after the nights sleep. At night one of the men got up and saw that if he gave one mango to the monkey, he could divide the rest of the mangos into three equal groups. So he ate his share and gave one mango to the monkey. Later a second person got up and he also noticed that if he gave one mango to the monkey he could divide the remaining mangos into three equal groups. So he gave one mango to the monkey and ate his share of the mangos and went to sleep. Finally the third man got up and gave one mango to the monkey and ate his share of the mangoes and went to sleep. When all of them got up in the morning, they again found that if they gave one mango to the monkey, they could divide the rest equally among themselves. The problem is to determine the smallest possible number of mangos that the men had bought.

The train had now started. My companion insisted that I occupy the upper berth while he would share his lower berth with two others. By the time the train arrived at the next station I was able to announce to my friend that the smallest number of mangos in the second problem was 79. Although we had no pen or paper, I could explain to him how I obtained the solution and he could verify the result. The next morning as the train steamed in at Allahabad, my friend woke me up and we parted as good friends. I was happy to have earned a friend by my effort to solve his problems.

PROBLEM 3. If you can find the day of the week from the date of birth of a person, you can make a good impression in any company and it becomes great fun to demonstrate this to the guests of the evening. This kind of problem is called a Calendar Problem.

We recall that the calendar that we use now is the Gregorian Calendar started by the Pope Paul Gregory in 1582 who fixed the days in the year as 365 but that every fourth year would be a leap year except when it is divisible by 400. Thus 1700, 1800, 1900 are not leap years, while 2000 is a leap year. If we keep in mind that the days of the week recur every 7^{th} day, all calculations in calendar problems are based on congruence modulo 7. (Recall that two integers are congruent modulo 7 if they have the same remainders when divided by 7.) This gives a convenient and easy formula for calculating the day of the week using the following conventions: the days of the week are numbered 0-6 beginning with Sunday, and the months are numbered 1-12 beginning with March.

This curious numbering is chosen because in each leap year February gets an extra day. Then February 28, 1999 will be considered as the last day of 1998. If someone is born on the r^{th} day of the m^{th} month of year N, set $N = 100C + D, 0 \leq D \leq 99,$ and we can obtain the day of the week by the following:

$$d = r + \left[\frac{13m-1}{5}\right] - 2C + D + \left[\frac{C}{4}\right] + \left[\frac{D}{4}\right] \pmod 7,$$

where $[x]$ = integral part of x and each term is expressed as *mod 7*. Let us calculate the day of the week for July 13, 1938. Here $r = 13$, $m = 5$, $C = 19$, $D = 38$, and so

$$\begin{aligned} d &= 13 + \left[\frac{13\cdot 5 - 1}{5}\right] - 2(19) + 38 \\ &\quad + \left[\frac{19}{4}\right] + \left[\frac{38}{4}\right] \pmod 7 \\ &= 13 + \left[\frac{64}{5}\right] - 38 + 38 \\ &\quad + \left[\frac{19}{4}\right] + \left[\frac{38}{4}\right] \pmod 7. \end{aligned}$$

Since $13 = 7 + 6$, we say that $13 = 6$ (mod 7), $\left[\frac{64}{5}\right] = 12 = 7 + 5 = 5$ (mod 7), $\left[\frac{19}{4}\right] = 4$, $\left[\frac{38}{4}\right] = 9 = 2$ (mod 7). Then

$$d = 6 + 5 + 4 + 2 = 17 = 3 \pmod 7.$$

Therefore July 13, 1938 falls on a Wednesday.

Secret Life of Professor Narc E. Sistic

Brooklynn Ann Welden

"Here you go, sir," the flight attendant said, handing the in-flight meal to the distinguished-looking man sitting next to the window, whose name was Dr. William Weighty. "How can you eat that," said his nearer companion, Dr. Rubric, tearing open a package of cookies, simultaneously sipping his soda without spilling. "Looks better than the cardboard you're about to ingest," remarked the third man, seated by the aisle, elbowing the flight attendant in his haste as he clicked open the overhead compartment and set his carry-on bag on his seat. The two men observed Dr. Lineynyi as he unzipped his bag and extracted a thick sandwich wrapped in plastic; both looked sour. "Made it myself," Lineynyi noted proudly.

Around them passengers ate and joked. Laughter rose from somewhere close by. Dr. Weighty, his mind on his upcoming mathematics conference, was in no mood to hear or even think about, Lineynyi's culinary triumph. Next to him, Rubric munched cookies as he browsed his freshly published article on wave propagation in porous environments. Back in his seat, Lineynyi extracted his worn copy of Milnor's Lectures on the h-cobordism theorem.

After the meal, the attendant reappeared, "Are you finished with that, sir?" Moving as one, Weighty, Rubric, and Lineynyi, held out their trash. The attendant found himself shifting the bag wildly, as he attempted to catch three pieces of trash dropped simultaneously from three hands in three different places. Stifling a sigh, he bent down and retrieved the mess from the aisle and the floor.

"So tell me, Rubric," said Weighty, noting with pleasure that he was interrupting the latter and wanting to relieve his own boredom, "what do you know about Professor Narc E. Sistic?" "Oh no," intoned Lineynyi, "the last time you talked about him, I got a stomachache from laughing. The man is a confounded embarrassment to us." "You mean he is an embarrassment to you," said Weighty. "Why, what happened?" asked Rubric, sticking his thumb into his book with anticipation, "Ive heard a lot of him—everyone I know has heard of him—but I haven't seen him for some time." "You miss nothing," opined Lineynyi. "You miss everything," contradicted Weighty, enjoying the expression of irritation on Lineynyi's face and feeling that they were approaching evenness regarding dinner. Rubric looked expectant; Lineynyi folded his arms and slid his feet forward as much as he could before folding himself forward until he was bent nearly in half. Weighty and Rubric regarded him in astonishment. 'He looks as though he were going to vomit,' Weighty thought, pleased at the prospect, calculating the angle of likely trajectory. Rubric, struck by the curve of Lineynyi's back, began pondering parabolas.

Lineynyi really should be telling this story, but as he doesn't seem eager, I will. Lineynyi was at a conference last month, listening to Prof. Narc E. Sistic. Now, when Dr. Narc E. Sistic talks, it is standing-room only; people are out the doors, in the aisles, and there are enough microphones and recording devices to stock the intelligence agency of a medium-sized country. Prof. Narc E. Sistic proved a theorem, and Lineynyi, who has been a rival of Sistic's for years, jumped up and said, "That proof must be wrong—I have a counterexample to your theorem." To which Prof. Narc E. Sistic replied, "I don't care—I have another proof for it."

Weighty and Rubric laughed uproariously and gave one another high-fives, startling several nearby passengers and causing one of them to yelp aloud in surprise. Lineynyi slid downward into his seat, arms folded so tightly he was in danger of self-suffocation. Observing Lineynyi's reaction, the pair laughed even harder, until tears streamed down their faces and they clutched at their

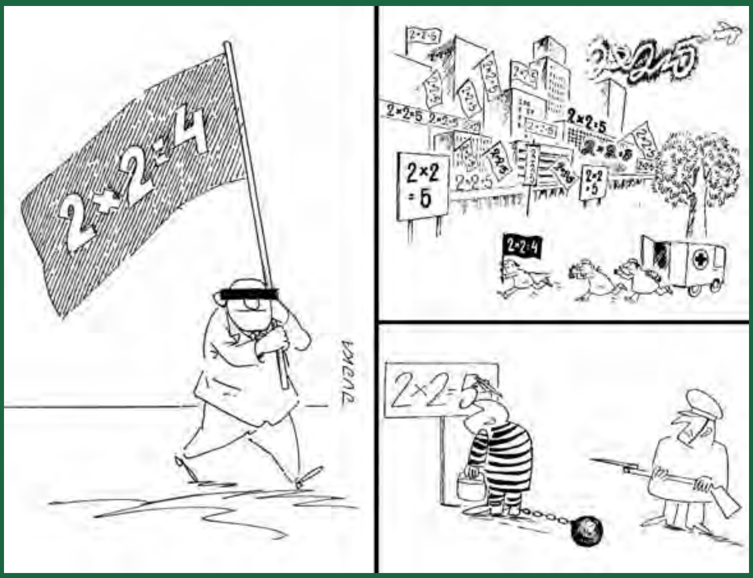
For a few minutes, the three mathematicians were silent. Gradually, Lineynyi's expression changed to one of crafty delight. He elbowed Rubric sharply in the ribs, making him grunt. "You want to hear more about Prof. Narc E. Sistic, eh, eh? Weighty there has a story to tell, but you'll never hear him mention it. I had it from a close, confidential, friend of his."

Lineynyi leaned back in his seat looking contented, while Rubric, who was rotund, inhaled eagerly and Weighty sank back in his seat, "How did you know about that?" "I said the person who told me was a confidential friend of yours who turned out to be a confidential friend of mine," Lineynyi intoned. Weighty, clearly knowing who the turncoat was, looked distressed, "Oh, Rubric doesn't want to hear about that." "Sure I do," said Rubric quickly, nearly breathless with excitement, "what happened?" "Well," said Lineynyi, "what I heard was that Professor Narc E. Sistic and Weighty here met at the races." "I never intended to bet," Weighty whined, interrupting, unsuccessfully trying to force the conversation onto a different track.". . . to bet on horses. Professor Narc E. Sistic suggests a bet of $10,000. Well, that's too much for old Weighty. So he tries to gather more information, and asks about the rules, wants a look at the horses, and so on. Professor Narc E. Sistic told Lineynyi not to worry, "I know an empirical algorithm that allows me to find the number of the winning horse with absolute certainty." Unfortunately, for him, Weighty was skeptical. "You are overly theoretical," Professor Narc E. Sistic told him. So Professor Narc E. Sistic puts $10,000 on a horse. And I don't need to tell you that of course the professor's horse won. Weighty was confounded, and demanded to know Professor Narc E. Sistic's algorithm. "That's rather easy," the professor said, "you have two children, three and five years old. I added up their ages and bet on that number." "But three plus five is eight, and that horse had number nine," Weighty shouted. "I did not shout," yelled Weighty. And Professor Narc E. Sistic said, "I told you that you are too theoretical. Didn't I just prove, experimentally, that my calculation is correct?"

This time Rubric and Lineynyi clutched one another and laughed until they cried, while Weighty looked sullenly out of the window. A few passengers, now twice startled by explosive laughter, appeared angry. "Oh, oh," laughed Rubric, holding onto his aching side. "Actually," said Rubric, once he and Lineynyi were calm again, "I have also had an experience with Professor Narc E. Sistic, one where I got the best of him." Weighty and Lineynyi both turned to look intently at Rubric, who, being younger than they, felt important. "Yes, I have the jump on you gentlemen as far as Professor Narc E. Sistic is concerned." Both Weighty and Lineynyi noted the lofty, superior tone Rubric adopted, with deepening displeasure. "My conversation with Professor Narc E. Sistic was short and I put him in his place. I met him quite by accident and he was expounding on his work in front of a circle of gasping, gaping admirers, mathematicians and students alike—you gentlemen know how Professor Narc E. Sistic is—when I asked him a question:" "'What is the difference between a mathematician and a philosopher?"' "Well, he stood there open-mouthed and silent, and all the students and mathematicians laughed and left, and Professor Narc E. Sistic was soon alone." Weighty and Lineynyi were suitably impressed; Rubric noticed the deference they now accorded him and preened.

A man, sitting in front of them and silent up until now, stood up and said, "I wish you loudmouths would keep quiet so we can sleep. That was not what happened; I was there. Professor Narc E. Sistic asked Rubric here the question, and Rubric did not know the answer." Weighty and Lineynyi, pleasantly surprised and relieved, burst out laughing once more, to a chorus of "shhh!" from irritated passengers. Rubric pulled out his book once more, pretending disinterest.

The flight attendant reappeared, looking stern, accompanied by another member of the flight crew with a threatening demeanor. Weighty and Lineynyi subsided quickly. From in front of them, a passenger, who had been listening to the entire conversation about Professor Narc E. Sistic with great interest, leaned toward the man who revealed Rubric's embarrassing secret, and asked, "So what was the answer to the question?" "Simple," the man replied, "I asked Professor Narc E. Sistic who told me that the mathematician only needs paper, pencil, and a trash bin for his work, but the philosopher can do without the trash bin."



Zbigniew Jujka

Problem Corner

Titu Andreescu

1. Set $a_0 = a_1 = 1$ and define
$$a_{n+1} = 1 + \frac{a_1^2}{a_0} + \cdots + \frac{a_n^2}{a_{n-1}}$$
for $n \geq 1$. Find a_n in closed form.

2. Let
$$P(x) = x^k + a_1 x^{k-1} + \ldots + a_k$$
be a non-constant polynomial with real coefficients. Evaluate
$$\lim_{n \to \infty} \tan \pi (P(n))^{\frac{1}{k}}.$$

3. Let k be a positive integer. Evaluate
$$\sum_{n=k+1}^{\infty} \frac{1}{n(n^2-1^2)\cdots(n^2-k^2)}.$$

Puzzle Corner: Plato's Stellar Predicament

Tobias Hagge

Plato's cosmology involves two fundamental motions. The first, called the *motion of the same*, was constant, unchanging, and eternal: the motion of perfect stars. The second, *motion of the other*, was composed of multiple parts, required to explain the more complicated apparent motion of planets (the *wanderers*). The star in Figure 13, however, has many parts. Using only your hands and with no additional folding, assemble the star into a small, *Stellated Dodecahedron* so that it may take its place in the sphere of fixed stars.

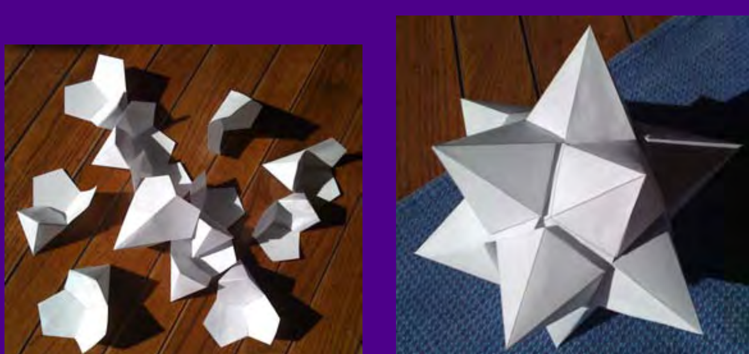


Figure 13: *The pieces and the complete dodecahedron.*

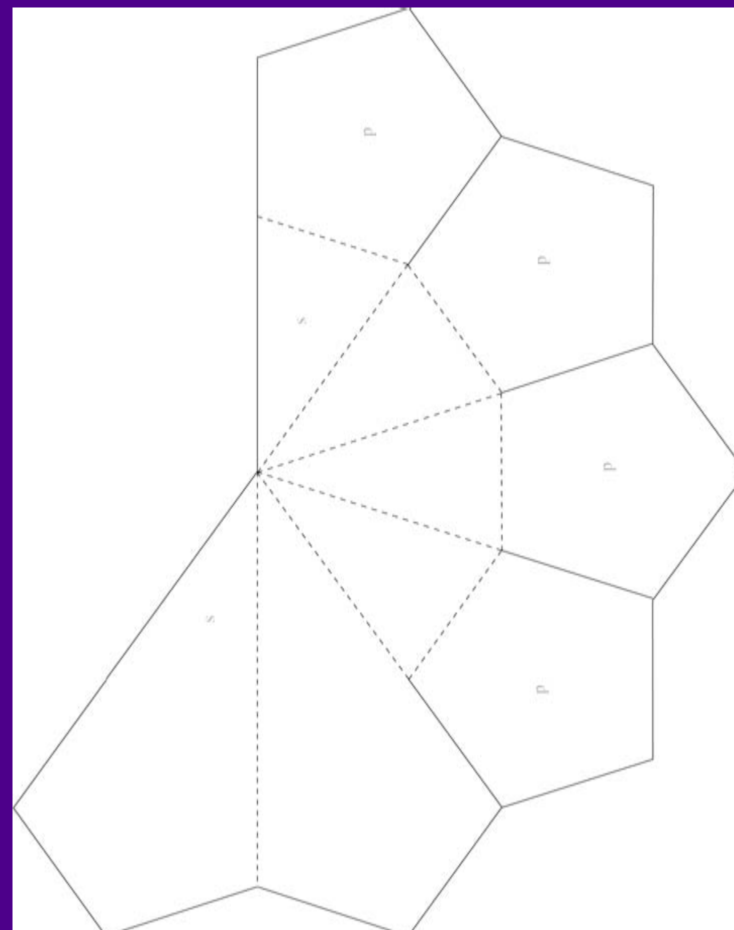


Figure 14: *Puzzle Template*

Building the pieces

Print the puzzle template[4] (Figure 14). Cut along the solid lines and fold along the dashed lines so that the lines are inside of the folds. Using a non-wrinkling glue such as scrapbooking glue or rubber cement, glue together the four faces labeled "p" and then the two faces labeled "s" to obtain the pieces shown in Figure 13.

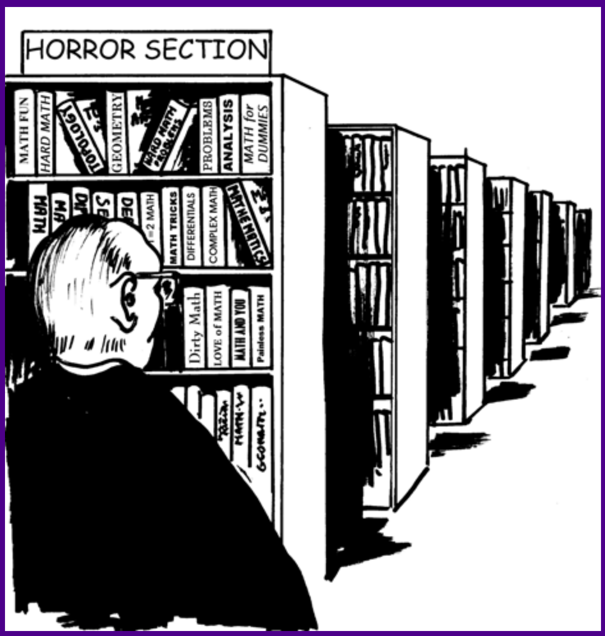
For a sturdier model, the template may be printed onto Cardstock or other thick paper, such as pasteboard. To fold Cardstock easily, use an Exacto (X-acto) Knife and a cork-backed steel ruler to lightly score the inside of each fold. Change blades frequently and cover old blades with masking tape. The blades are extremely sharp; watch your fingers!

Hints

Assemble the model on a non-slip surface such as a towel or tablecloth. Once assembled, the model holds together well enough to be moved. If desired, the pieces in the assembled model may be glued together to form a box with a removable lid.

Tobias Hagge, UTD

[4] You can download this template from the link:
http://www.utdallas.edu/~hagge/images/stellar_template.pdf

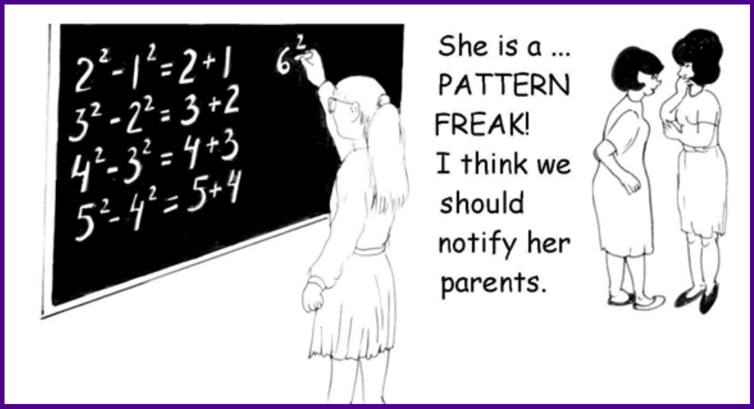


Wieslaw Z. Krawcewicz

Tasty Math Challenge Problems

Wieslaw Z. Krawcewicz

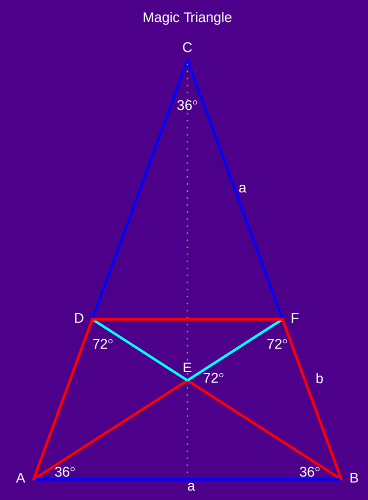
1. Use similar triangles to find the exact values of $\cos(\pi/5)$ and $\cos(2\pi/5)$.
   Hint. Recall that $\pi/5$ radians $= 36\,^\circ$. Draw an isosceles triangle with $72\,^\circ$ base angles. Many similar triangles can be formed from this *Magic Triangle*.
2. Find all the complex roots of the equation $z^5 = -1$.
3. Find the partial fraction decomposition of $\frac{1}{x^5+1}$.
4. Compute the indefinite integral
$$\int \frac{dx}{x^5+1}.$$



Wieslaw Krawcewicz

Solutions

1. To compute $\cos\pi/5$ and $\sin\pi/5$, we explore the similarities between the triangles generated from the triangle below.



$$a = \overline{AB} = \overline{AF} = \overline{CF}$$
$$b = \overline{DF} = \overline{FB} = \overline{EB}$$
$$\tfrac{\pi}{5} = 36\,^\circ$$

Notice that

$$\cos\frac{\pi}{5} = \frac{a/2}{b}.$$

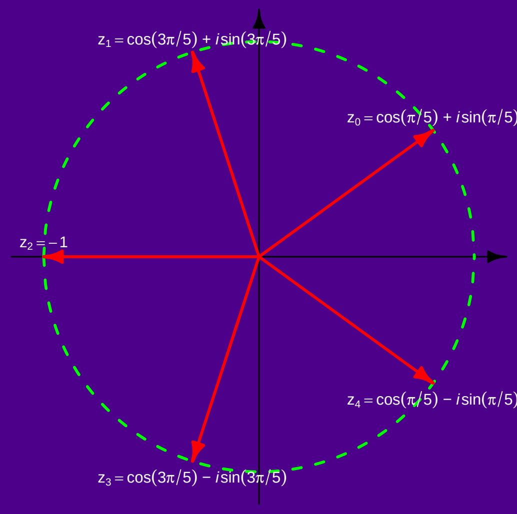
Since the triangles ABC and AFB are similar, $\frac{a+b}{a} = \frac{a}{b}$, which implies that $a^2 - ab - b^2 = 0$. Take, for example $a = 2$, then b can be determined from the quadratic equation $b^2 + 2b - 4 = 0$, which gives $b = \sqrt{5} - 1$. Therefore, $\cos\frac{\pi}{5} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}$. and

$$\sin\frac{\pi}{5} = \sqrt{1-\cos^2\frac{\pi}{5}} = \sqrt{1-\frac{6+2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

2. By using the trigonometric form $-1 = e^{i\pi}$, one finds that

$$z_k = \cos\left(\frac{\pi+2\pi k}{5}\right) + i\sin\left(\frac{\pi+2\pi k}{5}\right),\quad k = 0,1,2,3,4,$$

are the roots of the equation $z^5 = -1$.



The evaluated values of $\cos\frac{\pi}{5}$ and $\sin\frac{\pi}{5}$ allow us to compute z_0,

z_1, z_2, z_3, and z_4:

$$\begin{aligned}
z_0 &= \frac{1+\sqrt{5}}{4} + i\frac{\sqrt{10-2\sqrt{5}}}{4}\\
z_1 &= z_0^3 = \frac{1-\sqrt{5}}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}\\
z_2 &= -1\\
z_3 &= \frac{1-\sqrt{5}}{4} - i\frac{\sqrt{10+2\sqrt{5}}}{4}\\
z_4 &= \frac{1+\sqrt{5}}{4} - i\frac{\sqrt{10-2\sqrt{5}}}{4}
\end{aligned}$$

where $z_4 = \overline{z_0}$ and $z_3 = \overline{z_0}$.

3. First we consider the complex polynomial $P(z) = z^5 + 1$. By the Fundamental Theorem of Algebra, $P(z)$ can be completely factorized as

$$\begin{aligned}
P(z) &= (z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)\\
&= (z+1)(z-z_0)((z-\overline{z_0})(z-z_1)(z-\overline{z_1})
\end{aligned}$$

Since

$$(z-w)(z-\overline{w}) = z^2 - \text{Re}(w)z + |w|^2),$$

we obtain

$$\begin{aligned}
z^5+1 &= (z+1)(z^2 - 2\text{Re}(z_0)z + 1)(z^2 - 2\text{Re}(z_1)z + 1)\\
&= (z+1)\left(z^2 - \frac{1+\sqrt{5}}{2}z + 1\right)\left(z^2 - \frac{1-\sqrt{5}}{2}z + 1\right)
\end{aligned}$$

Thus, the required factorization is

$$x^5+1 = (x+1)\left(x^2 - \frac{1+\sqrt{5}}{2}x + 1\right)\left(x^2 - \frac{1-\sqrt{5}}{2}x + 1\right).$$

With this factorization we now can find the complex partial fraction decomposition:

$$\frac{1}{z^5+1} = \frac{A_0}{z-z_0} + \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \frac{A_3}{z-z_3} + \frac{A_4}{z-z_4}.$$

After putting all the fractions over the common denominator $z^5 + 1$ and comparing the numerators, one gets the equality

$$1 = \sum_{k=0}^{4} A_k \frac{(z-z_0)(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{z-z_k},$$

which can be written as

$$1 = \sum_{k=0}^{4} A_k \frac{P(z)}{z-z_k}.$$

Thus, for $m = 0, 1, 2, 3, 4,$

$$\begin{aligned}
1 &= \lim_{z\to z_m} \sum_{k=0}^{4} A_k \frac{P(z)}{z-z_k} = \sum_{k=0}^{4} A_k \lim_{z\to z_m} \frac{P(z)-P(z_m)}{z-z_k}\\
&= A_m P'(z_m) = 5A_m z_m^4,
\end{aligned}$$

which gives us

$$A_k = \frac{1}{5z_k^4} = -\frac{1}{5}z_k.$$

Then for $k = 0$ and $k = 1$ we have

$$\frac{z_k}{z-z_k} + \frac{\overline{z_k}}{z-\overline{z_k}} = \frac{2\text{Re}(z_k)z - 2}{z^2 - 2\text{Re}(z_k)z + 1}$$

which gives us the following partial fraction decomposition

$$\frac{1}{x^5+1} = \frac{1}{5}\left[\frac{1}{x+1} + \frac{2 - \frac{1+\sqrt{5}}{2}x}{x^2 - \frac{1+\sqrt{5}}{2}x + 1} + \frac{2 - \frac{1-\sqrt{5}}{2}x}{x^2 - \frac{1-\sqrt{5}}{2}x + 1}\right].$$

4. Notice that the integral

$$I := \int \frac{2 - 2\cos\theta x}{x^2 - 2\cos\theta x + 1}dx$$

can be evaluated by applying the substitution $u = x - \cos\theta$, i.e.

$$\begin{aligned}
I &= 2\sin^2\theta \int \frac{du}{u^2 + \sin^2\theta} - \cos\theta \int \frac{2udu}{u^2+\sin^2\theta}\\
&= 2\sin\theta \arctan\left(\frac{x-\cos\theta}{\sin\theta}\right) - \cos\theta \ln(x^2 - 2\cos\theta x + 1) + C
\end{aligned}$$

Since

$$\begin{aligned}
&\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}, \quad \sin\frac{\pi}{4} = \frac{\sqrt{10-2\sqrt{5}}}{4},\\
&\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}, \quad \sin\frac{3\pi}{4} = \frac{\sqrt{10+2\sqrt{5}}}{4},
\end{aligned}$$

we obtain that

$$\begin{aligned}
\int \frac{1}{x^5+1}dx =& \frac{1}{5}\ln|x+1|\\
&+ \frac{\sqrt{10-2\sqrt{5}}}{10} \arctan \frac{4x-1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\\
&- \frac{1+\sqrt{5}}{20} \ln\left(x^2 - \frac{1+\sqrt{5}}{2}x + 1\right)\\
&+ \frac{\sqrt{10+2\sqrt{5}}}{10} \arctan \frac{4x-1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}\\
&- \frac{1-\sqrt{5}}{20} \ln\left(x^2 - \frac{1-\sqrt{5}}{2}x + 1\right)
\end{aligned}$$

CHALLENGE: Apply the above method to find an explicit formula for the integral

$$\int \frac{1}{x^n+1}dx, \quad \text{where } n = 2, 3, 4, \ldots$$

The Elusive and Omnipresent Nature of Randomness

Arturo Sangalli

The theory of probability is the best tool we humans have devised to deal with random phenomena–that is, those situations involving chance. The simplest example of a random event is the result of flipping a *balanced* or *fair* coin, which can land either heads or tails. If we repeatedly toss the coin ten times and record each successive outcome by writing 1 (for heads) and 0 (for tails) we end up with a sequence of 0s and 1s (i.e. a binary sequence) such as, for example, **0 0 0 1 1 0 1 1 0 1**. We would then say that the sequence was obtained by a random process, since chance alone decided its composition rather than a predetermined rule.

We would perhaps also be tempted to say that the above sequence is a random sequence, since it resulted from a random process. But what if we had obtained **0 1 0 1 0 1 0 1 0 1** or **0 0 0 0 0 0 0 0 0 0** instead? Such sequences look anything but random, and yet they too could have been randomly generated. In fact, any binary sequence of length 10 is equally likely to occur (with probability $1/2^{10}$) as a result of tossing the coin ten times in succession, but they cannot all be *random*, for then the notion of randomness would be meaningless. At this point, the attentive reader would probably like to stop me and ask: "Wait a second, what do you mean by a sequence being *random*?" The point of my article is precisely to (try) to answer this question.

Actually, it is fairly easy to put in words what a random binary sequence is: one that is completely unpredictable, that obeys no rule. It is not that we are not smart enough to figure out the rule or pattern, but that there is no rule, and in this sense true randomness is indistinguishable from absolute chaos. The tricky part is to come up with a definition that would capture those features.

For example, sequences such as **0 1 0 1 0 1 0 1 0 1** or **0 1 1 0 1 1 1 0 0 1 0 1** should not qualify as random, for we can predict what the next bit will be, assuming **...** implies the pattern continues. In the first case, 0s and 1s alternate, so the 11^{th} bit will certainly be 0, the 12^{th} will be 1, and so on. The other sequence is constructed by writing the numbers **0, 1, 2, 3, 4, 5,** in base 2, one after the other, i.e. **0, 1, 10, 11, 100, 101**, etc, so that we would know what bit must appear in position $n+1$ by looking at the first n bits.

Several definitions of "random sequence" were proposed in the first half of the 20^{th} century, but none of these proved satisfactory. In 1919, the Austrian-born mathematician Richard von Mises came up with the following definition: an infinite sequence **S** of 0s and 1s is random if

(a) **S** satisfies the law of large numbers (LLN), that is, "the proportions of 0s and 1s are the same in the long run", or, more precisely,

$$\lim_{n\to\infty} x_n/n = 0.5,$$

where x_n is the number of 0s among the first n terms of the sequence,

(b) every subsequence that can be extracted from **S** by *reasonable means* also satisfies the LLN.

Application of von Mises definition to the sequence **0 1 0 1 0 1 0 1** of alternating 0s and 1s would confirm that it is not random, for the subsequence of even bits (the 2^{nd}, 4^{th}, etc.) would be a sequence of all 1s, and so would not satisfy the LLN. Likewise, many other sequences which appear intuitively to be non-random fail to satisfy von Mises conditions, and hence they are not random also in this technical sense.

Unfortunately, the definition proposed by von Mises suffered from a fundamental flaw: it did not specify which means for extracting a subsequence are "reasonable" means. To remedy this situation, the American mathematician Alonzo Church suggested in 1940 that condition (b) of von Mises definition should apply only to computable subsequences–that is, to those subsequences whose terms could be defined by a computer program. Although Churchs idea had the merit of making the definition precise, examples were subsequently found of sequences that are intuitively non-random but nevertheless satisfy the von Mises-Church notion of *randomness*. The modified definition was therefore too large and the concept of a binary random sequence still refused to be pinned down.

Finally, in 1965, Per Martin-Löf, a young Swedish mathematician, came up with a sophisticated definition that appears to be the right one. His definition can be better understood through its connection with a concept developed by the Russian mathematician Andrej Kolmogorov. In the 1970s, Kolmogorov defined the *complexity* of a finite binary sequence as the length of the shortest computer program that can print it. For example, a sequence of one million 1s has very small complexity, for there are very short programs that can be used to print it. For instance, in many computer languages something of the form:

```
for i = 1 to 1,000,000, print 1
```

would produce this sequence. Such a program may be considered a *compressed* version of the given sequence. On the other hand, a sequence of 1,000,000 bits whose Kolmogorov Complexity is at least 1,000,000 is totally *incompressible*: there is no shorter way of describing it than listing all its one million bits.

Now, *randomness* and *incompressibility* turned out to be equivalent concepts, meaning that: an infinite binary sequence is random (in Martin-Löfs sense) precisely if it is incompressible. An immediate practical consequence of this result is that no computer program can generate a truly random binary sequence. The reason is simple: any sequence **S** whose bits are obtained by executing a computer program is compressible by definition. It follows that all random number algorithms being used in computer programs are in fact only *pseudo-random*, not simply for lack of ingenuity on the part of the mathematicians who created them, but due to the existence of an essential barrier inherent in the concept of *randomness* itself.

The situation is made even more puzzling by the fact that, even if *almost all* binary sequences are random, no algorithm or computer program can produce a single one of them. Randomness is everywhere and yet it is out of reach! Or is it? In 1974, Gregory Chaitin, a mathematician working at IBM, came up with a real number between 0 and 1 he called Ω. When written in base 2, the digits of Ω form a true random sequence. In order to understand where Ω came from we need to make a brief detour through the theory of computation.

In the summer of 1935, a young Cambridge graduate reflecting on a question in the foundations of mathematics introduced the notion of an ideal computer that could mimic the operations of any real computing device. The young graduate was Alan Turing – the brilliant British mathematician behind the cracking of the German Enigma code during World War II, Turing is considered the father of the modern computer. His ideal computing machine became known as a Turing Machine.

Figure 15: *Alan Turing Memorial, Manchester, UK*

Since Turing's ideal machine is at least as powerful as any real one, anything a Turing Machine cannot do, no real computer - present or future - will be able to do either. As Turing showed, one of the problems his ideal computer cannot solve concerns the automatic checking of computer software. This is the question of determining in advance whether any given computer program, when executed, will eventually terminate its calculation and halt or is destined to run forever – the so-called *halting problem*.

Starting in the 1970s, Chaitin took a fresh interest in the halting problem. He considered all the programs that could be run on a Turing Machine and asked the following question: *What is the probability that one of these programs chosen at random will halt?* He found that the answer is the number he called Ω. The digits of Ω are an example of a true random sequence, in the sense that they have no pattern or structure whatsoever. Chaitin described Ω as a string of 0s and 1s in which each digit is unrelated to its predecessors as one coin toss is from the next, and presented Ω as the outstanding example of something in mathematics that is uncomputable, and therefore unknowable.

But Chaitin did not stop there. He began to search for other places in mathematics where randomness might crop up, and he found that it does in its most elementary branch: arithmetic. But if there was randomness at the most basic mathematical level, his hunch was that it must be everywhere, that randomness is the true foundation of mathematics. In Chaitin's own words: "God not only plays dice in physics but also in pure mathematics. Mathematical truth is sometimes nothing more than a perfect coin toss."

If Chaitin's hunch is correct, the implications are far-reaching. It means that we may be able to prove some theorems, answer some questions, but the vast majority of mathematical problems are essentially unsolvable. It means that a few bits of math may follow from one another, but for most mathematical situations those connections will not exist, because mathematics is full of accidental, reasonless truths. And if you can't make connections, you can't solve problems–or prove things. If Chaitin's intuition is right, solvable problems would be like a small island in a vast sea of undecidable propositions.

And so our attempt at understanding randomness took us in unexpected directions, from a simple coin toss to fundamental questions in the philosophy of mathematics - such is the elusive and omnipresent nature of the concept.

This article is based on material from the author's book **Pythagoras Revenge: A Mathematical Mystery**, Princeton University Press, 2009.

Reasonable Doubt

Larry P. Ammann

Preamble of the U.S. Constitution

We the People of the United States, in Order to form a more perfect Union, establish Justice, insure domestic Tranquility, provide for the common defence, promote the general Welfare, and secure the Blessings of Liberty to ourselves and our Posterity, do ordain and establish this Constitution for the United States of America.

Judicial decision-making is one of the fundamental duties of any government, and the principles underlying its implementation are crucially important to the success and longevity of a government. Yet, it seems that most people in the U.S. have only a vague understanding of the principles that guide our judicial system, including the meaning of reasonable doubt, the central component of this system. Since the logic behind the principles of judicial decision-making is the same as the logic associated with statistical hypothesis testing, we often discuss judicial trials in our Statistics courses as a way to introduce hypothesis testing. What I would like to present here is a more detailed discussion of the principles of judicial decision-making and their implementation in our system of justice. This discussion will consider those issues associated with the process that governs all trials, not those that arise from a particular individual trial, although there will be some aspects of these principles that could add insight to some of the more controversial trials that have been conducted in our recent history. This discussion is presented from the perspective of a professor of Statistics, not a professor of Constitutional Law. Readers are encouraged to seek out other discussions of these issues from experts in that field.

When someone is arrested and charged with a criminal offense, a decision must be made about the guilt or innocence of the accused. A policy governing the process that is followed to arrive at this decision must address three basic attributes of any system for judicial decision-making:

1. the risk of convicting an innocent defendant
2. the risk of not convicting a guilty defendant
3. time allowed to make a decision

The first two attributes can be represented by the following table:

		Actual State: Innocent	Actual State: Guilty
Decision	Innocent	√	Error
	Guilty	Error	√

Two possible errors can occur, depending on the decision. If the jury's decision is not to convict, but the defendant is actually guilty, then an error has occurred. This error, in which a guilty defendant is set free, is something we don't want to have happen. The thought of a criminal being set free after committing a terrible crime is very disturbing. While it is possible to design a policy under which guilty defendants are never set free, the only possibility is a policy which convicts every defendant. As this also guarantees that every innocent defendant would be convicted, the policy is unreasonable. A different type of error occurs if the jury's decision is to convict, but the defendant is actually innocent. We don't want this error to occur either, especially if we are the innocent defendants who are convicted under that policy. The only way to guarantee that never happens is not to convict anyone. But as this would allow every guilty defendant to be set free, it also is an unreasonable policy. Therefore, the only sensible policy for judicial decision-making is one under which, from time to time, innocent defendants are convicted and from time to time, guilty defendants are not convicted.

Sixth Amendment to the U.S. Constitution

In all criminal prosecutions, the accused shall enjoy the right to a speedy and public trial by an impartial jury of the State and district wherein the crime shall have been committed, which district shall have been previously ascertained by law, and to be informed of the nature and cause of the accusation; to be confronted with the witnesses against him; to have compulsory process for obtaining witnesses in his favor, and to have the Assistance of Counsel for his defense.

Since we must design a policy under which both types of errors may occur, it would be desirable for this policy to guarantee that these mistakes happen very infrequently. However, there is a third component to judicial decision-making that makes this impossible: the time needed to make the decision. Because these three attributes of judicial decision-making are interrelated, we can control only two of the three attributes; the third is a function of the other two. Therefore, the only way to ensure that both types of errors occur very infrequently is not to place any boundary on the time required to reach a decision. For example, it is very likely that some crimes more than 25 years old could have been adjudicated correctly if the trials for those crimes had waited for today's DNA identification techniques. We can reasonably expect that 25 years from now there will be new technologies available that would allow correct adjudication of some current trials if we deferred making those decisions until that technology becomes available. The problem associated with having no set boundary to the time allowed to make a decision is how to treat the defendant while waiting for that future technology. If the defendant is imprisoned during this time, the effect would be just the same as though the

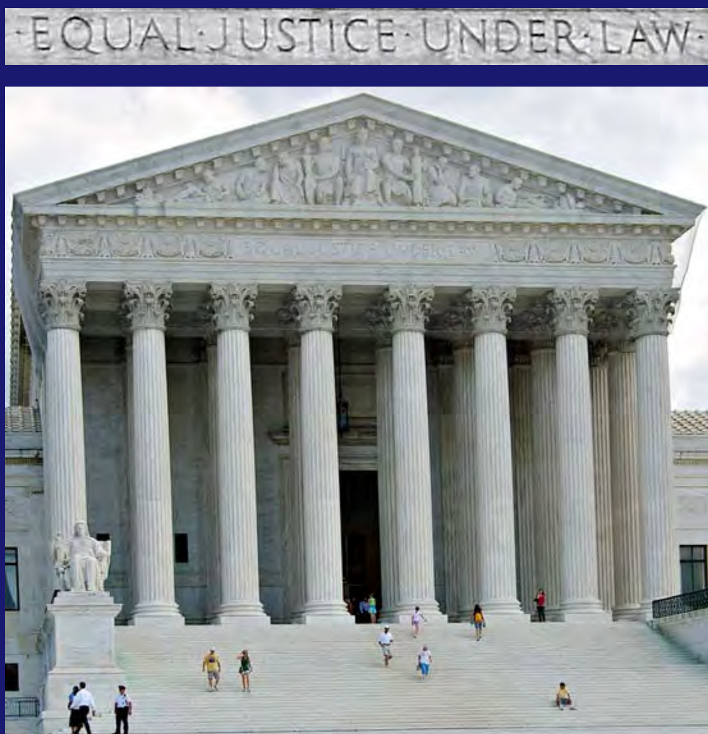


Figure 16: *U.S. Supreme Court Building*

defendant were convicted and sentenced to at least 25 years in prison. If the defendant is not imprisoned, the effect would be identical with not convicting the defendant. Therefore, the Sixth Amendment places a limit on the amount of time allowed between arrest and trial.

Fifth Amendment to the U.S. Constitution

No person shall be held to answer for a capital, or otherwise infamous crime, unless on a presentment or indictment of a Grand Jury, except in cases arising in the land or naval forces, or in the Militia, when in actual service in time of War or public danger; nor shall any person be subject for the same offense to be twice put in jeopardy of life or limb; nor shall be compelled in any criminal case to be a witness against himself, nor be deprived of life, liberty, or property, without due process of law; nor shall private property be taken for public use, without just compensation.

Fourteenth Amendment to the U.S. Constitution, Section 1

All persons born or naturalized in the United States, and subject to the jurisdiction thereof, are citizens of the United States and of the State wherein they reside. No State shall make or enforce any law which shall abridge the privileges or immunities of citizens of the United States; nor shall any State deprive any person of life, liberty, or property, without due process of law; nor deny to any person within its jurisdiction the equal protection of the laws.

We are left with a choice: design a policy under which time to make a decision is limited and the rate of convicting innocent defendants is kept low, or design a policy under which time to make a decision is limited and the rate of not convicting guilty defendants is kept low. The choice encoded in our Constitution is to require that the time to make a decision is limited and the rate of convicting innocent defendants is kept low. To implement this policy, juries may only convict defendants if they determine that the evidence is very strong for guilt. That is, a jury may convict only if it believes that the doubt concerning their decision to convict is reasonably small. This is the Common Law interpretation of the clause, due process of law, in the Fifth and Fourteenth Amendments. If the evidence points towards innocence, if it is inconclusive, or if it is only somewhat strong for guilt, then the jury cannot convict the defendant. Likewise, the jury cannot convict if no evidence is presented. This implies that only the hypothesis of guilt is required to be proven, not innocence. That is, the burden of proof in a trial is placed on the decision to convict.

Since the error which occurs when innocent defendants are convicted is the error we are concerned about and whose rate of occurrence we want kept small, that error is referred to as a Type I error. The other error, in which a guilty defendant is not convicted, is referred to as a Type II error.

		Actual State	
		Innocent	Guilty
Decision	Innocent	√	Type II error
	Guilty	Type I error	√

There is a direct consequence of placing the burden of proof onto the decision to convict. Since we cannot believe a defendant is guilty until after the jury makes a decision to convict, then prior to that time we must presume that the defendant is innocent and treat the defendant accordingly. In particular, this implies that a person who is arrested is not a criminal until a jury returns a guilty verdict, and therefore is entitled to the rights of all persons as defined by the Fifth, Sixth, and Fourteenth Amendments: the right to remain silent, the right to an attorney, the right to a speedy and public trial, etc. This also implies that the strength of the evidence for guilt that is presented during a trial must be assessed under the presumption of the defendant's innocence. That assessment can be thought of as an answer to the question, "how likely is it that the incriminating evidence presented by the prosecution could accrue against an innocent defendant?" This likelihood represents the juror's assessment of the strength of the evidence for guilt. It also represents the likelihood that an innocent defendant has been convicted if the jury decides to convict based on the evidence of the trial.

To follow the due process clause of the Fifth and Fourteenth Amendments, each juror must determine his or her definition of reasonably small doubt. This represents what the juror considers to be acceptably small for the risk of convicting an innocent defendant. When put into the context of all trials held under our system of justice, the definition of reasonably small doubt represents what a juror considers to be reasonably small for the proportion of innocent defendants who are wrongly convicted for the crimes in question. A decision to convict or not convict is made by comparing the definition of reasonable doubt to the juror's assessment of the strength of evidence for guilt. If the juror's assessment of the strength of evidence for guilt does not satisfy his or her definition of reasonably small doubt, then the decision must be: do not convict. This decision would be reported as: the evidence is not sufficiently strong to prove guilt. If the juror's assessment of the evidence meets or is stronger than the definition of reasonably small doubt, then the decision should be to convict. In that case, the likelihood that the evidence has accrued against an innocent defendant represents the chance the jury would be making a Type I error. Note that if the evidence is not sufficiently strong to convict, this does not imply that innocence was proven.

The standard for reasonable doubt should depend on the consequences associated with a Type I error. For example, if a defendant is on trial for simple misdemeanor theft and the punishment is a $500 fine, then the consequences of a Type I error in this case would be that an innocent person would have to pay this fine unnecessarily and would have a police record of this conviction. If a defendant is on trial for capital murder, then the consequence of Type I error could be that the State would execute an innocent person. Clearly, our standards for reasonably small doubt should be different in these cases.

There is one aspect of judicial decision-making that is not present in statistical decision-making. Reasonable doubt is defined in terms of what is considered to be acceptable for the proportion of innocent defendants who are convicted. If the experiences of some jurors are such that innocent defendants are arrested and brought to trial relatively infrequently, then those jurors would not be inclined to set an extremely stringent standard for reasonable doubt. To such jurors, a moderately small risk of convicting an innocent defendant would still guarantee that few innocent defendants would be convicted since, in their experience, only a relatively small number of innocent defendants are brought to trial. However, other jurors may have experiences that have convinced them that the number of innocent people who have been arrested is not small. Those jurors would be expected to set a much stronger standard for reasonable doubt to ensure that the number of innocent defendants who are convicted is kept reasonably small. Those different experiences could account at least partially for cultural differences in the reactions to some recent sensational trials such as the O.J. Simpson trial.

One conclusion that can be made from this discussion is that we must make every effort to ensure that the process of arresting and charging individuals with crimes does not result in the arrest of innocent people very often. To achieve that, the collection and presentation of evidence for trials must not be abused. All of the rights granted by our Constitution must be strictly enforced and applied to all who have been charged but not yet convicted of crimes. Finally, impartial review of evidence and trial procedures must be available for defendants who have been convicted. If any abuse of these processes is tolerated and influences the outcomes of a few selected trials, then it would be natural to expect that such abuses would continue to affect other trials. Ultimately, faith in our system of justice would be seriously eroded.

Note: the opinions and interpretations presented here are solely and entirely those of the author. Readers are encouraged to explore these and related issues on their own. An excellent resource for such investigations is the Library of Congress, and a visit to the Library is highly recommended to anyone who plans to visit Washington, D.C.

Figure 17: *Library of Congress*

Afterword

Putting together this first issue has been a labor of love by faculty members of the Department of Mathematical Sciences at the University of Texas at Dallas. Mathematics is a language that has made possible the progress of our civilization and is crucial for finding solutions to current and future challenges to our welfare and survival. But Mathematics also can be fun and artistic. It is our goal to illustrate these aspects of the language in addition to some of its more serious applications.

Editorial Notes

Wieslaw Z. Krawcewicz

The article "Solving Problems Can Be Fun" was written by my friend at the University of Alberta, Ambikeshwar Sharma (1920-2003). The article was first published in "Pi in the Sky" magazine, which I was publishing for the Pacific Institute for Mathematical Sciences at that time. Sharma was an expert in the theory of interpolation, a man deeply devoted to his profession. He was also a charismatic and inspiring teacher. I will always remember him. One cartoon in this issue of "Connexion" is the work of Zbigniew Jujka—a Polish cartoonist from Gdansk who was famous for resisting communist power. The other cartoons are mine.

Back Cover

Larry P. Ammann

Modern digital photography and mathematics are inseparable. A digital image is just a collection of numbers that represent brightness levels of a set of fundamental colors. Conversion of light energy recorded by a camera's sensor into a photographic image requires sophisticated mathematical operations referred to as image processing. Future issues of this magazine will include articles about Photographic Image Processing. The back covers will be devoted to photographs of people, places, and activities around the DFW Metroplex.

The back cover of this issue is part of a series of photographs I took at White Rock Lake in Dallas. From late fall through early spring, White Rock Lake provides R&R for numerous ducks, geese, white pelicans, and other waterfowl. White pelicans in particular are very expressive birds. One day last year I was taking photos of these beautiful birds taking off from the east shore of White Rock. To get airborne on days with little or no wind, pelicans use a combination of wing flapping and hopping with their feet to overcome their large size. Right after shooting one such series, a Great White Egret landed next to a pelican. The egret then took off, effortlessly becoming airborne with just one flap of its wings. When I processed those shots, I noticed that the pelican was watching the takeoff, and so I imagined what she must have thought about that egret.

Photo credits. Solar eclipse: NASA; Moon: L.P. Ammann; Saturn: Cassini Imaging Team, SSI, JPL, ESA, NASA; M20: Adam Block/Mt. Lemmon SkyCenter/U. Arizona; M13: STScI POSS-II DSS, N. Carboni; M31: B. Schoening, V. Harvey/REU/NOAO/AURA/NSF; Cerro Tololo: NOAO; M51: Adam Block/Mt. Lemmon SkyCenter/U. Arizona; M104: Adam Block/Mt. Lemmon SkyCenter/U. Arizona; M77: Adam Block/Mt. Lemmon SkyCenter/U. Arizona; NGC 5394: Adam Block/Mt. Lemmon SkyCenter/U. Arizona; U. Allahabad: W. Krawcewicz; Alan Turing Memorial: GNU FDL; U.S. Supreme Court: L.P. Ammann; Library of Congress: D. Iliff.

Department of Mathematical Sciences

Mathematical Sciences at UTD includes both graduate and undergraduate degree programs. The undergraduate program offers Bachelor of Science degrees in Actuarial Science, Mathematics, Applied Mathematics, and Statistics. The graduate program offers Master of Science degrees in Mathematics, Applied Mathematics, Engineering Mathematics, and Statistics. An interdisciplinary degree program in Bioinformatics and Computational Biology is jointly offered by Math Sciences and the Department of Molecular and Cell Biology. Math Sciences also offers Ph.D. degrees in Applied Mathematics and Statistics. Advisors for our degree programs are:

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Janos Turi, UTD

Mieczyslaw Dabkowski, UTD

[Bleep]ing Anorexic Egret Showoff!