Vector Calculus Review Sheet

1 Gradient and Directional Derivatives

The del operator ∇ is defined as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{1}$$

The **gradient** of a scalar function $f(x, y, z)$, written $\operatorname{grad} f$ or ∇f, is the vector function defined as

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{2}$$

If $\operatorname{grad} f$ is non-zero at some point P, then it points in the direction of the greatest increase of f at P.

If f is interpreted as the scalar function that represents some surface $S : f(x, y, z) = c$ where c is some constant, then $\operatorname{grad} f$ evaluated some point P on S is the normal of S at P. The unit surface normal $\mathbf{n}$ to S is thus

$$\mathbf{n} = \frac{1}{|\operatorname{grad} f|} \operatorname{grad} f \tag{3}$$

Some useful formulas for grad are

$$\nabla(f^n) = nf^{n-1}\nabla(f) \tag{4}$$

$$\nabla(fg) = f\nabla(g) + g\nabla(f) \tag{5}$$

$$\nabla(f/g) = (1/g^2)(g\nabla(f) - f\nabla(g)) \tag{6}$$

The **directional derivative** $D_{\mathbf{a}}$ of some scalar function f in the direction of vector $\mathbf{a}$ is given by:

$$D_{\mathbf{a}}f = \frac{1}{|\mathbf{a}|}\, \mathbf{a} \cdot \operatorname{grad} f \tag{7}$$

If $\operatorname{grad} v = \mathbf{E}$, then the function v is called the **potential function** (or simply potential) of $\mathbf{E}$.

Conversely, a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is called a **gradient field** if there exists a scalar function f such that $\operatorname{grad} f = \nabla f = \mathbf{F}$. If this is the case, then the form $F_1\, dx + F_2\, dy + F_3\, dz$ is said to be **exact**.

2 Divergence and Curl

Given a vector field $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, we define the **divergence** of the field as

$$\operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \tag{8}$$

and the **curl** of the field as

$$\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{9}$$

Note that the divergence is a scalar quantity while the curl is a vector quantity.

The divergence can be thought of as the flux of the field through a tiny (i.e., differential) volume element. The curl can be thought of as the measure of the rotational velocity of the field.

The **Laplacian** of a scalar function f, written $\nabla^2 f$, is the divergence of the gradient of f, i.e.,

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{10}$$

Useful formulas for div and curl are

$$\operatorname{div}(f\mathbf{v}) = f\operatorname{div}\mathbf{v} + \mathbf{v}\cdot\nabla f \tag{11}$$
$$\operatorname{div}(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g \tag{12}$$
$$\operatorname{div}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\operatorname{curl}\mathbf{u} - \mathbf{u}\cdot\operatorname{curl}\mathbf{v} \tag{13}$$
$$\operatorname{curl}(f\mathbf{v}) = \nabla f\times\mathbf{v} + f\operatorname{curl}\mathbf{v} \tag{14}$$
$$\operatorname{curl}(\nabla f) = \mathbf{0} \tag{15}$$
$$\operatorname{div}(\operatorname{curl}\mathbf{v}) = 0 \tag{16}$$

A useful formula for the Laplacian is

$$\nabla^2(fg) = g\nabla^2 f + 2\nabla f\cdot\nabla g + f\nabla^2 g \tag{17}$$

A vector field $\mathbf{v}$ is said to be **incompressible** if and only if $\operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} = 0$.

A vector field $\mathbf{v}$ is said to be **irrotational** if and only if $\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = 0$.

3 Line Integrals

Let $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ be some vector function. Further, let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be the parameterization of some curve C from $a = \mathbf{r}(t_0)$ to $b = \mathbf{r}(t_1)$.

The **line integral** of $\mathbf{F}(\mathbf{r})$ along C is defined as

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt \tag{18}$$

Setting $\mathbf{F}(x(t), y(t), z(t)) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ and evaluating the dot product, we see that

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_x\, dx + F_y\, dy + F_z\, dz) = \int_{t_0}^{t_1} \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} \right) dt \tag{19}$$

Given a scalar function $f(x, y, z)$, the line integral of this function along C is defined as

$$\int_C f(\mathbf{r})\, dt = \int_{t_0}^{t_1} f(\mathbf{r}(t))\, dt \tag{20}$$

where $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$.

The notation $\oint$ implies that the line integral is taken across a closed path.

The line integral of a vector function $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is said to be **path independent** if and only if F_x, F_y, and F_z are continuous in a domain D, and if there exists some scalar function f in D such that $\mathbf{F} = \operatorname{grad} f$. If this is the case, then the line integral of $\mathbf{F}$ along a curve C from A to B is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_A^B (F_1\, dx + F_2\, dy + F_3\, dz) = f(B) - f(A) \tag{21}$$

Hence, if $\mathbf{F}$ is path independent,

$$\oint \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \mathbf{0} \tag{22}$$

The line integral of a vector function $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is path independent if and only if

$$\operatorname{curl} \mathbf{F} = \mathbf{0} \tag{23}$$

Note that equation (23) is satisfied, then the form $F_1\, dx + F_2\, dy + F_3\, dz$ is exact.

4 Change of Variables

In certain instances, it is easier to integrate in a non-Cartesian coordinate system (e.g., polar, spherical, or cylindrical coordinates). When doing so, it is necessary to use a "fudge factor" when performing the integration.

For **polar coordinates** (in 2-D), recall that $x = r\cos\theta$ and $y = r\sin\theta$. The double integral in Cartesian coordinates (i.e., in terms of x and y) becomes

$$\iint_R f(x, y)\, dx\, dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\; r\, dr\, d\theta \tag{24}$$

where R^* is the region in the $r\theta$-plane that corresponds to R in the xy-plane.

For **cylindrical coordinates**, recall that $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. The triple integral in Cartesian coordinates becomes

$$\iiint f(x, y, z)\, dx\, dy\, dz = \iiint f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz \tag{25}$$

For **spherical coordinates**, recall that $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$. The triple integral in Cartesian coordinates becomes

$$\iiint f(x, y, z)\, dx\, dy\, dz = \iiint f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\, \rho^2\sin\phi\;\, d\rho\, d\theta\, d\phi \tag{26}$$

5 Surfaces and Surface Integrals

The general form of a parametric representation of a surface in space is

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

The circular cylinder $x^2 + y^2 = a^2,\ -b \le z \le b$, has radius a, height $2b$, and has the z-axis as its axis. Its parameterization is

$$\mathbf{r}(u,v) = a\cos u\,\mathbf{i} + a\sin u\,\mathbf{j} + v\mathbf{k} \tag{27}$$

where $0 \le u \le 2\pi,\ -b \le v \le b$.

The sphere $x^2 + y^2 + z^2 = a^2$ can be parameterized either as

$$\mathbf{r}(u,v) = a\cos v\cos u\,\mathbf{i} + a\cos v\sin u\,\mathbf{j} + a\sin v\,\mathbf{k} \tag{28}$$

where $0 \le u \le 2\pi,\ -\pi/2 \le v \le \pi/2$, or as

$$\mathbf{r}(u,v) = a\cos u\sin v\,\mathbf{i} + a\sin v\sin u\,\mathbf{j} + a\cos v\,\mathbf{k} \tag{29}$$

where $0 \le u \le 2\pi,\ 0 \le v \le \pi$.

If $\mathbf{r}(u,v)$ is the parameterization of a surface S, then the normal vector $\mathbf{N}$ to S at point P is given by the formula

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \ne \mathbf{0} \tag{30}$$

where $\mathbf{r}_u = \partial\mathbf{r}/\partial u$ and $\mathbf{r}_v = \partial\mathbf{r}/\partial v$ are the partial derivatives of $\mathbf{r}$ with respect to u and v. From this, we see that the surface normal vector $\mathbf{n}$ is

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\,\mathbf{r}_u \times \mathbf{r}_v \tag{31}$$

Given a piecewise smooth surface S parameterized $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, the **flux** or **surface integral** of a vector function $\mathbf{F}$ though S is defined as

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\mathbf{A} = \iint_R \mathbf{F}[\mathbf{r}(u,v)]\cdot\mathbf{N}(u,v)\,du\,dv \tag{32}$$

where $\mathbf{N}$ is the surface normal to S defined in equation 30 and $\mathbf{n}$ is the unit surface normal. Here, R is the region in the uv-plane that corresponds to S.

If we let $\mathbf{F} = F_x(u,v)\mathbf{i} + F_y(u,v)\mathbf{j} + F_z(u,v)\mathbf{k}$ and $\mathbf{N} = N_x(u,v)\mathbf{i} + N_y(u,v)\mathbf{j} + N_z(u,v)\mathbf{k}$, then we can write the surface integral as

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\mathbf{A} = \iint_R \left(F_xN_x + F_yN_y + F_zN_z\right)\,du\,dv \tag{33}$$

Note that the surface integral in equation (32) is evaluated in reference to a particular orientation of $\mathbf{N}$. Thus, replacing $\mathbf{N}$ with $-\mathbf{N}$ has the effect of multiplying the integral in equation (32) by -1.

We can define another type of surface integral that is evaluated without regard to orientation:

$$\iint_S G(\mathbf{r})\,dA = \iint_R G(\mathbf{r}(u,v))\,|\mathbf{N}(u,v)|\,du\,dv \tag{34}$$

In this integral, G represents the mass density function of S; note that if $G = 1$, evaluation of this integral gives the total (surface) area of S.

6 Integral Theorems

Green's theorem: Let R be a region in the xy-plane that is bounded by the closed, piecewise smooth curve C. Let $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$ be a continuous vector function with continuous first partial derivatives in a some domain containing R. Then,

$$\iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\, dy = \oint_C (F_x\, dx + F_y\, dy) \tag{35}$$

or equivalently,

$$\iint_R (\text{curl}\, \mathbf{F}) \cdot \mathbf{k}\, dx\, dy = \oint_C \mathbf{F} \cdot d\mathbf{r} \tag{36}$$

Gauss' divergence theorem: Let T be a closed region bounded by a surface S and let $\mathbf{F}(x, y, z)$ be a continuous vector function that has continuous first partial derivatives in some domain containing T. Then,

$$\iiint_T \text{div}\, \mathbf{F}\, d\text{V} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\text{A} \tag{37}$$

Stoke's theorem: Let S be a piecewise smooth surface and let the boundary of S be a simple, piecewise smooth curve C. Let $\mathbf{F}(x, y, z)$ be a continuous vector function that has continuous first partial derivatives in some domain containing S. Then,

$$\iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, d\text{A} = \oint_C \mathbf{F} \cdot \mathbf{r}'(s)\, ds \tag{38}$$

Note that Green's theorem is simply Stoke's Theorem applied to a 2-dimensional plane.