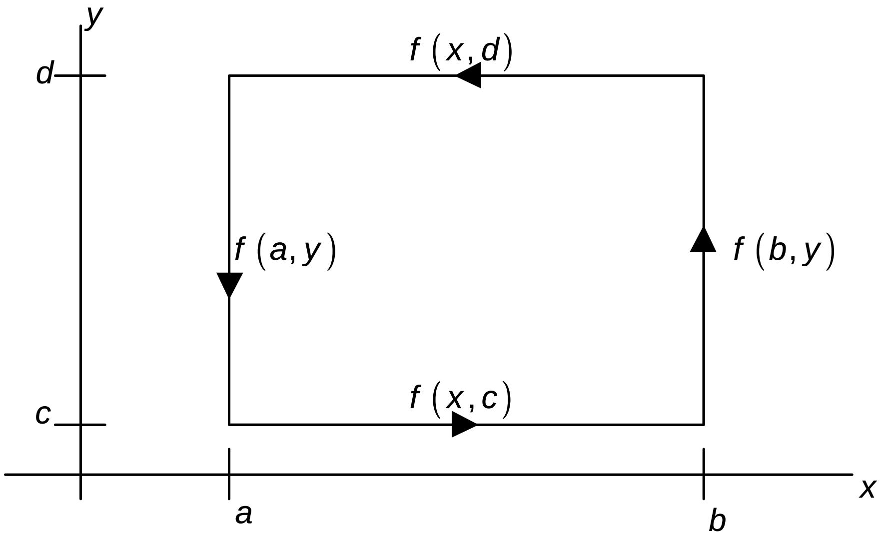
Prove Green's Theorem:



The total line integral of $f(x,y)$ around the chosen path in the x-y plane is the sum of the line integral of $f(x,y)$ around the chosen path with respect to x and the line integral of $f(x,y)$ around the chosen path with respect to y:

$$\oint_C \left[f(x,y)\,dx + f(x,y)\,dy \right] = \oint_C f(x,y)\,dx + \oint_C f(x,y)\,dy$$

$$\oint_C f(x,y)\,dx = \int_a^b f(x,c)\,dx + \int_b^a f(x,d)\,dx = \int_a^b f(x,c)\,dx - \int_a^b f(x,d)\,dx = \int_a^b f(x,c) - f(x,d)\,dx$$

$$\oint_C f(x,y)\,dy = \int_c^d f(b,y)\,dy + \int_d^c f(a,y)\,dy = \int_c^d f(b,y)\,dy - \int_c^d f(a,y)\,dy = \int_c^d f(b,y) - f(a,y)\,dy$$

From Calculus, the two line integrals can be written as:

$$\int_a^b\int_c^d \frac{\partial f\left(x,y\right)}{\partial y}dydx = \int_a^b f\left(x,d\right) - f\left(x,c\right)dx$$

$$\int_a^b\int_c^d \frac{\partial f\left(x,y\right)}{\partial x}dydx = \int_c^d f\left(b,y\right) - f\left(a,y\right)dy$$

Therefore,

$$\oint_C \left[f\left(x,y\right)dx + f\left(x,y\right)dy\right] = \oint_C f\left(x,y\right)dx + \oint_C f\left(x,y\right)dy = \int_a^b f\left(x,c\right) - f\left(x,d\right)dx + \int_c^d f\left(b,y\right) - f\left(a,y\right)dy$$

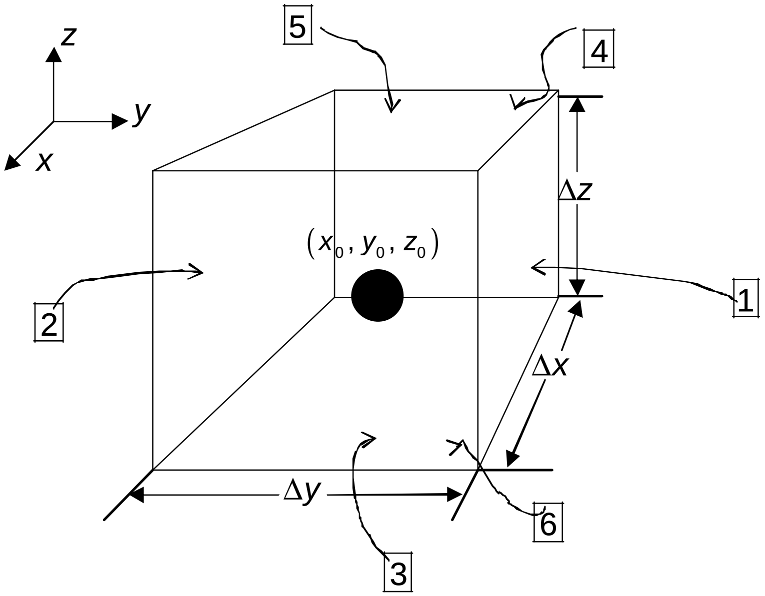
$$= -\int_a^b f\left(x,d\right) - f\left(x,c\right)dx + \int_c^d f\left(b,y\right) - f\left(a,y\right)dy$$

$$= -\int_a^b\int_c^d \frac{\partial f\left(x,y\right)}{\partial y}dydx + \int_a^b\int_c^d \frac{\partial f\left(x,y\right)}{\partial x}dydx$$

$$= \int_a^b\int_c^d \left[\frac{\partial f\left(x,y\right)}{\partial x} - \frac{\partial f\left(x,y\right)}{\partial y}\right]dydx$$

$$\boxed{\oint_C \left[f\left(x,y\right)dx + f\left(x,y\right)dy\right] = \iint_R \left[\partial_x f - \partial_y f\right]dA}$$

Prove the Divergence Theorem:



The total flux of and arbitrary vector field $\mathbf{V}(x, y, z)$ emanating out of the closed box can be written as:

$$f_{total} = \oint_S \mathbf{V} \cdot d\mathbf{s} = \int_S \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$$

Expanding the vector flux integral into its individual components:

$$f_{total} = \left[\int_1 \mathbf{V}_1 \cdot \hat{\mathbf{y}} \, dxdz + \int_2 \mathbf{V}_2 \cdot \left(-\hat{\mathbf{y}}\right) dxdz \right] + \left[\int_3 \mathbf{V}_3 \cdot \hat{\mathbf{x}} \, dydz + \int_4 \mathbf{V}_4 \cdot \left(-\hat{\mathbf{x}}\right) dydz \right] + \left[\int_5 \mathbf{V}_5 \cdot \hat{\mathbf{z}} \, dxdy + \int_6 \mathbf{V}_6 \cdot \left(-\hat{\mathbf{z}}\right) dxdy \right]$$

Evaluate the vector field $\mathbf{V}$ on each surface:

$$\left. \begin{array}{l} \mathbf{V}_1(x, y, z) \Rightarrow \mathbf{V}_1\left(x, y_0 + \dfrac{\Delta y}{2}, z\right) \\ \mathbf{V}_2(x, y, z) \Rightarrow \mathbf{V}_2\left(x, y_0 - \dfrac{\Delta y}{2}, z\right) \end{array} \right\} \text{integration with respect to } x \text{ and } z$$

$$\left.\begin{array}{l}\mathbf{V}_3(x,y,z) \Rightarrow \mathbf{V}_3\left(x_0+\dfrac{\Delta x}{2},y,z\right)\\ \mathbf{V}_4(x,y,z) \Rightarrow \mathbf{V}_4\left(x_0-\dfrac{\Delta x}{2},y,z\right)\end{array}\right\}\text{integration with respect to } y \text{ and } z$$

$$\left.\begin{array}{l}\mathbf{V}_5(x,y,z) \Rightarrow \mathbf{V}_5\left(x,y,z_0+\dfrac{\Delta z}{2}\right)\\ \mathbf{V}_6(x,y,z) \Rightarrow \mathbf{V}_6\left(x,y,z_0-\dfrac{\Delta z}{2}\right)\end{array}\right\}\text{integration with respect to } y \text{ and } x$$

Take Pairs 1 and 2 and Taylor Expand them:

$$\mathbf{V}_1\left(x,y_0+\frac{\Delta y}{2},z\right)=\mathbf{V}(x,y_0,z)+\partial_y\mathbf{V}(x,y_0,z)\left(\frac{\Delta y}{2}\right)+\partial_y^2\mathbf{V}(x,y_0,z)\left(\frac{\Delta y}{2}\right)^2+\text{smaller terms}$$

$$\mathbf{V}_2\left(x,y_0-\frac{\Delta y}{2},z\right)=\mathbf{V}(x,y_0,z)+\partial_y\mathbf{V}(x,y_0,z)\left(-\frac{\Delta y}{2}\right)+\partial_y^2\mathbf{V}(x,y_0,z)\left(-\frac{\Delta y}{2}\right)^2+\text{smaller terms}$$

Subtract 2 from 1, because in the integral we have a $(\mathbf{V}_1-\mathbf{V}_2)\square\mathbf{y}$ term:

$$\mathbf{V}_1\left(x,y_0+\frac{\Delta y}{2},z\right)-\mathbf{V}_2\left(x,y_0-\frac{\Delta y}{2},z\right)=\partial_y\mathbf{V}(x,y_0,z)\left(\frac{\Delta y}{2}\right)+\partial_y\mathbf{V}(x,y_0,z)\left(\frac{\Delta y}{2}\right)=\partial_y\mathbf{V}(x,y_0,z)\Delta y$$

Now let Δy become infinitesimal:

$$\lim_{\Delta y\to 0}\left[\mathbf{V}_1\left(x,y_0+\frac{\Delta y}{2},z\right)-\mathbf{V}_2\left(x,y_0-\frac{\Delta y}{2},z\right)\right]=\lim_{\Delta y\to 0}\left[\partial_y\mathbf{V}(x,y_0,z)\Delta y\right]=\partial_y\mathbf{V}(x,y_0,z)\,dy$$

Similarly, we obtain for the other pairs:

$$\lim_{\Delta x\to 0}\left[\mathbf{V}_3\left(x_0+\frac{\Delta x}{2},y,z\right)-\mathbf{V}_4\left(x_0-\frac{\Delta x}{2},y,z\right)\right]=\lim_{\Delta x\to 0}\left[\partial_x\mathbf{V}(x_0,y,z)\Delta x\right]=\partial_x\mathbf{V}(x_0,y,z)\,dx$$

$$\lim_{\Delta z\to 0}\left[\mathbf{V}_5\left(x,y,z_0+\frac{\Delta z}{2}\right)-\mathbf{V}_6\left(x,y,z_0-\frac{\Delta z}{2}\right)\right]=\lim_{\Delta z\to 0}\left[\partial_z\mathbf{V}(x,y,z_0)\Delta z\right]=\partial_z\mathbf{V}(x,y,z_0)\,dz$$

Plugging back into the original flux integral:

$$f_{total} = \int \lim_{\Delta y \to 0}\left[\mathbf{V}_1\left(x, y_0 + \frac{\Delta y}{2}, z\right) - \mathbf{V}_2\left(x, y_0 - \frac{\Delta y}{2}, z\right)\right] dxdz + \int \lim_{\Delta x \to 0}\left[\mathbf{V}_3\left(x_0 + \frac{\Delta x}{2}, y, z\right) - \mathbf{V}_4\left(x_0 - \frac{\Delta x}{2}, y, z\right)\right] dydz + \int \lim_{\Delta z \to 0}\left[\mathbf{V}_5\left(x, y, z_0 + \frac{\Delta z}{2}\right) - \mathbf{V}_6\left(x, y, z_0 - \frac{\Delta z}{2}\right)\right] dydx$$

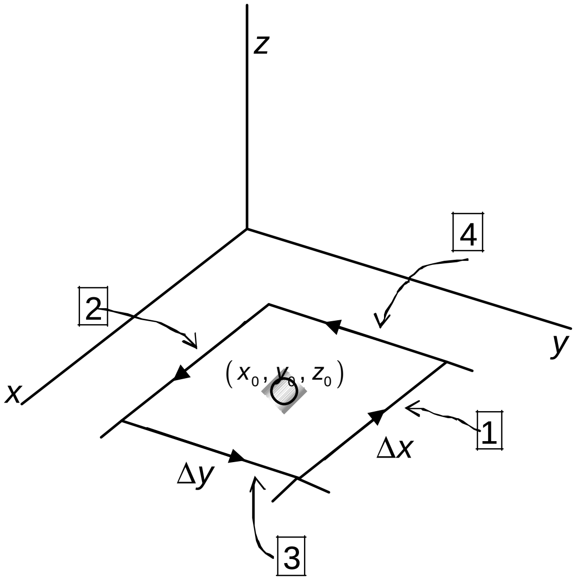
$$f_{total} = \oiint_S \mathbf{V} \cdot d\mathbf{s} = \int \partial_y \mathbf{V} dydxdz + \int \partial_x \mathbf{V} dxdydz + \int \partial_z \mathbf{V} dzdydx$$

$$f_{total} = \oiint_S \mathbf{V} \cdot d\mathbf{s} = \int \partial_x \mathbf{V} + \partial_y \mathbf{V} + \partial_z \mathbf{V} dxdydz$$

$$f_{total} = \oiint_S \mathbf{V} \cdot d\mathbf{s} = \int \left(\partial_x + \partial_y + \partial_z\right) \cdot \mathbf{V} dxdydz$$

$$\boxed{f_{total} = \oiint_S \mathbf{V} \cdot d\mathbf{s} = \int_V \nabla \cdot \mathbf{V} dV}$$

Prove Stokes' Theorem:



The line integral of an arbitrary vector field $\mathbf{V}(x, y, z)$ around the defined closed loop in the x-y plane can be expanded into its individual components:

$$\oint_C \mathbf{V}\cdot d\mathbf{l} = \left[\int_1 \mathbf{V}_1\cdot\left(-\hat{\mathbf{x}}\right)dx + \int_2 \mathbf{V}_2\cdot\hat{\mathbf{x}}\,dx\right] + \left[\int_3 \mathbf{V}_3\cdot\hat{\mathbf{y}}\,dy + \int_4 \mathbf{V}_4\cdot\left(-\hat{\mathbf{y}}\right)dy\right]$$

Evaluate the vector field on the line segments:

$$\left.\begin{array}{l} \mathbf{V}_1(x, y, z) \Rightarrow \mathbf{V}_1\left(x, y_0 + \dfrac{\Delta y}{2}, z_0\right) \\ \mathbf{V}_2(x, y, z) \Rightarrow \mathbf{V}_2\left(x, y_0 - \dfrac{\Delta y}{2}, z_0\right) \end{array}\right\}\text{integration with respect to } x$$

$$\left.\begin{array}{l} \mathbf{V}_3(x, y, z) \Rightarrow \mathbf{V}_3\left(x_0 + \dfrac{\Delta x}{2}, y, z_0\right) \\ \mathbf{V}_4(x, y, z) \Rightarrow \mathbf{V}_4\left(x_0 - \dfrac{\Delta x}{2}, y, z_0\right) \end{array}\right\}\text{integration with respect to } y$$

Taylor Expand Pair 1 and 2:

$$\mathbf{V}_1\left(x,y_0+\frac{\Delta y}{2},z_0\right)=\mathbf{V}(x,y_0,z_0)+\partial_y\mathbf{V}(x,y_0,z_0)\left(\frac{\Delta y}{2}\right)+\partial_y^2\mathbf{V}(x,y_0,z_0)\left(\frac{\Delta y}{2}\right)^2+\text{smaller terms}$$

$$\mathbf{V}_2\left(x,y_0-\frac{\Delta y}{2},z_0\right)=\mathbf{V}(x,y_0,z_0)+\partial_y\mathbf{V}(x,y_0,z)\left(-\frac{\Delta y}{2}\right)+\partial_y^2\mathbf{V}(x,y_0,z_0)\left(-\frac{\Delta y}{2}\right)^2+\text{smaller terms}$$

Subtract 1 from 2 because we have a $(\mathbf{V}_2-\mathbf{V}_1)\cdot\hat{\mathbf{x}}$ term in the integral:

$$\mathbf{V}_2\left(x,y_0-\frac{\Delta y}{2},z_0\right)-\mathbf{V}_1\left(x,y_0+\frac{\Delta y}{2},z_0\right)=-\partial_y\mathbf{V}\Delta y$$

Now letting Δy become infinitesimal, we shrink the line segments so that $\mathbf{V}\Rightarrow V_x$, i.e. the vector field becomes the x-component of the vector field in the limit as Δy approaches 0:

$$\lim_{\Delta y\to 0}\left[\mathbf{V}_2\left(x,y_0-\frac{\Delta y}{2},z_0\right)-\mathbf{V}_1\left(x,y_0+\frac{\Delta y}{2},z_0\right)\right]=-\partial_y V_x dy$$

Similarly, we obtain for Pair 3 and 4:

$$\lim_{\Delta x\to 0}\left[\mathbf{V}_3\left(x_0+\frac{\Delta x}{2},y,z\right)-\mathbf{V}_4\left(x_0-\frac{\Delta x}{2},y,z\right)\right]=\partial_x V_y dx$$

Plugging these back into the original line integral:

$$\oint_C \mathbf{V}\cdot d\mathbf{l}=\int -\partial_y V_x dydx+\int \partial_x V_y dxdy$$

$$\oint_C \mathbf{V}\cdot d\mathbf{l}=\int\left(\partial_x V_y-\partial_y V_x\right)dxdy$$

$$\oint_C \mathbf{V}\cdot d\mathbf{l}=\int(\nabla\times\mathbf{V})_z\cdot\hat{\mathbf{z}}dxdy$$

$$\boxed{\oint_C \mathbf{V}\cdot d\mathbf{l}=\iint_S(\nabla\times\mathbf{V})\cdot\hat{\mathbf{n}}dA}$$

Physical Interpretation of the Divergence Theorem:

$$\boxed{\oint_S \mathbf{V}\cdot d\mathbf{s} = \int_V \nabla\cdot\mathbf{V}\,dV}$$

The total outward flux of the vector field $\mathbf{V}$ emanating from a closed surface is equal to the sum of the contributions of all the point sources (of flux) in the volume enclosed by the surface. Think of $\nabla\cdot\mathbf{V}$ as a measurement of the source or sink of the vector field. Recall from Advanced Engineering Math that if $\nabla\cdot\mathbf{V} = 0$, then we call $\mathbf{V}$ "*incompressible.*"

Physical Interpretation of Stoke's Theorem:

$$\boxed{\oint_C \mathbf{V}\cdot d\mathbf{l} = \oint_S \left(\nabla\times\mathbf{V}\right)\cdot d\mathbf{s}}$$

The total circulation of the vector field $\mathbf{V}$ around a closed contour C is equal to the sum of the contributions of all the point sources (of vector rotation) on the surface area bounded by the contour C. Think of $\nabla\times\mathbf{V}$ as a measurement of the rotational properties of the vector field $\mathbf{V}$. Recall from Advanced Engineering Math that if $\nabla\times\mathbf{V} = \mathbf{0}$, then we call $\mathbf{V}$ "*irrotational.*"

You will see the power of these theorems later on in Emag when we take Maxwell's Equations from the familiar Integral Form to "Point Form."

Curves and Surfaces:

Curves can be parameterized by a single variable, for instance, t. Curves can sit inside $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$. If the curve is $\mathbb{R}^n$, then the parameterization will read as $x_1 = a_1(t), x_2 = a_2(t), \ldots, x_n = a_n(t)$ for specific functions $a_1(t), a_2(t), \ldots, a_n(t)$. We can then form the parametric representation $\mathbf{r}(t) = (a_1(t), a_2(t), a_3(t), \ldots, a_n(t))$.

The tangent to the curve then becomes $\mathbf{r}'(t)$.

Likewise, surfaces can be parameterized by two variables s and t. Surfaces can sit inside $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$. A surface inside $\mathbb{R}^n$ can be parameterized by n functions: $x_1 = a_1(s,t), x_2 = a_2(s,t), \ldots, x_n = a_n(s,t)$. We can then form the parametric representation $\mathbf{r}(s,t) = (a_1(s,t), a_2(s,t), a_3(s,t), \ldots, a_n(s,t))$.

Since there are two variables, there are two tangents to the surface given by $\partial_s \mathbf{r}(s,t)$ and $\partial_t \mathbf{r}(s,t)$.

The normal to the surface is simply the cross product of these two tangents: $\mathbf{n} = \partial_t \mathbf{r}(s,t) \times \partial_s \mathbf{r}(s,t)$

Parameterization of Commonly Used Curves:

1. *Circle in* $\mathbb{R}^2$
   Let $\mathbf{r}(t) = (r\cos(t), r\sin(t))$, where the radius r is fixed and the parameter t varies as $t \in [0, 2\pi]$.

2. *Circle in* $\mathbb{R}^3$
   Let $\mathbf{r}(t) = (r\cos(t), r\sin(t), c)$ where the radius r is fixed, t varies as $t \in [0, 2\pi]$, and c is some constant that indicates the z-plane in which the circle sits.

3. *Helix*
   Let $\mathbf{r}(t) = (r\cos(t), r\sin(t), t)$ for $t \in [0, 2\pi]$.

4. *Ellipse in* $\mathbb{R}^2$

   Let $\mathbf{r}(t) = (a\cos(t), b\sin(t))$ for $t \in [0, 2p]$, where a and b are the constants in the equation of the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

5. *Parabola*

   Let $\mathbf{r}(t) = (t, t^2)$ for $t \in [-\infty, \infty]$.

6. *Straight line in* $\mathbb{R}^2$ *between an initial point* (a, b) *and a final point* (a, b)

   Let $\mathbf{r}(t) = (a + t(a - a), b + t(b - b))$ for $t \in [0, 1]$.

7. *Triangles, Squares, etc.*

   Use 6 to parameterize each side of the object.

8. *Straight line through an initial point* (a, b, c) *and a final point* (a, b, g)

   Let $\mathbf{r}(t) = (a + t(a - a), b + t(b - b), c + t(g - c))$ for $t \in [0, 1]$.

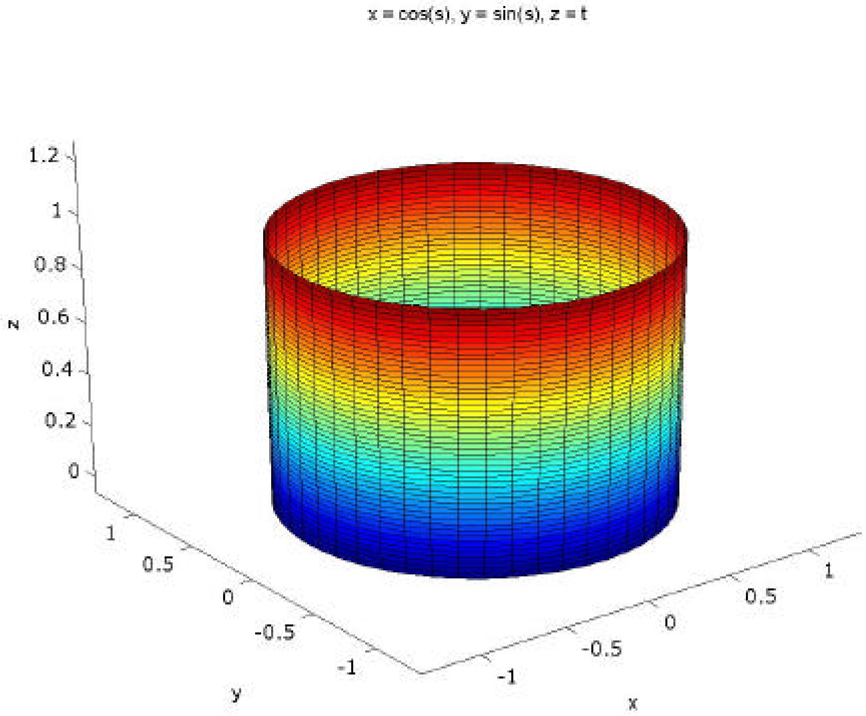
Parameterizations of Commonly Used Surfaces:

1. *Graph of a function* $f(x,y)$ *of 2 variables.*

   Let $\mathbf{r}(s,t) = (s,t,f(s,t))$

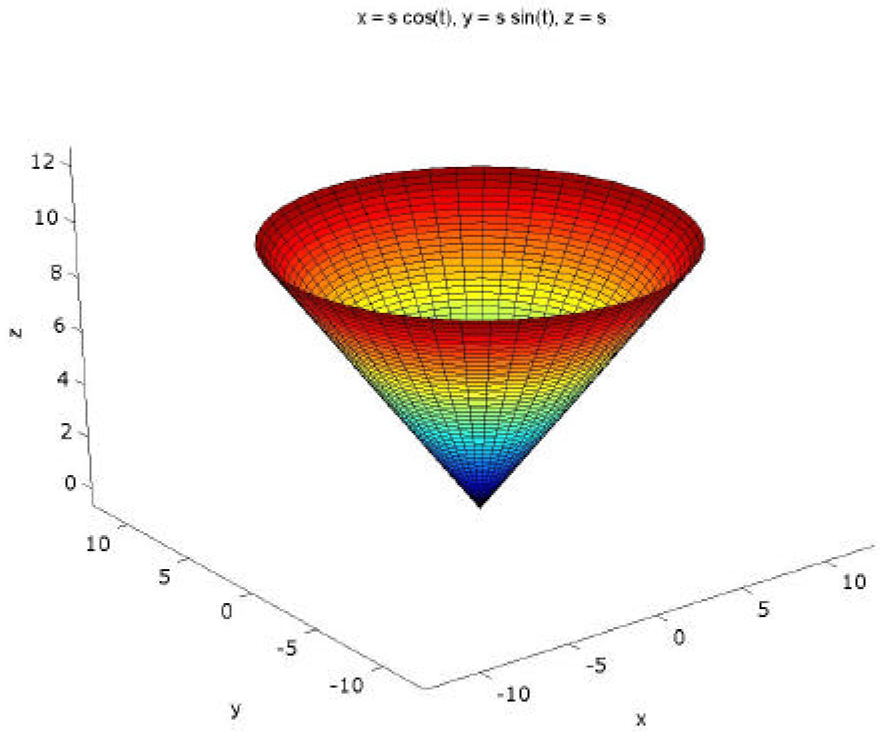
2. *Cylinder*

   Let $\mathbf{r}(s,t) = (r\cos(s), r\sin(s), t)$ for $s \in [0, 2p]$ and $t \in [0, L]$.



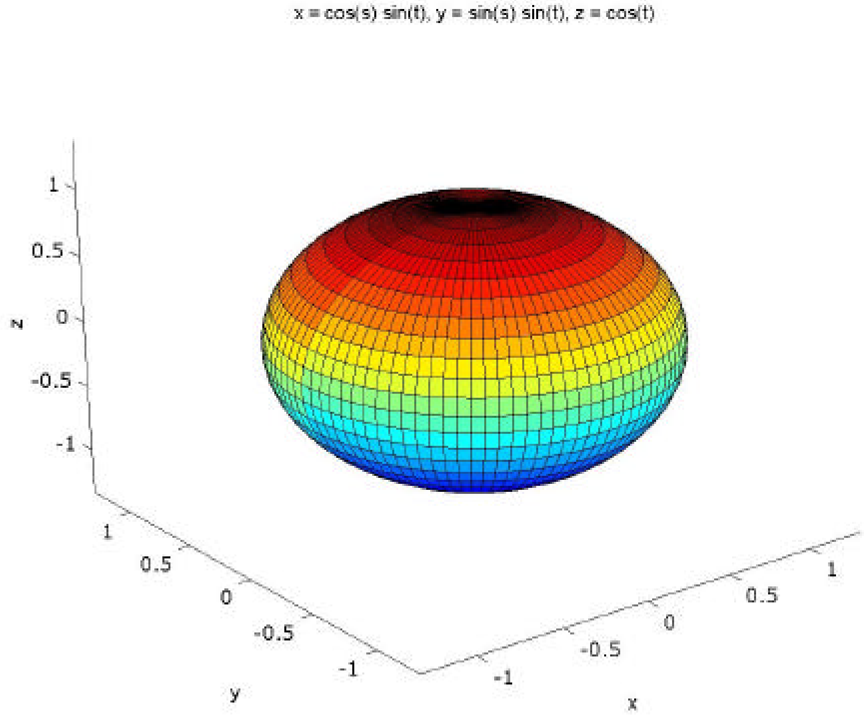
3. *Cone of vertex* $(0,0,0)$ *and axis of symmetry, the* z *-axis*

   Let $\mathbf{r}(s,t) = (s\cos(t), s\sin(t), s)$ for $s \geq 0$ and $t \in [0, 2\pi]$



4. *Sphere of radius* r, *centered at* $(0,0,0)$

$$\mathbf{r}(s,t) = \left(r\cos(f)\sin(q),\, r\sin(f)\sin(q),\, r\cos(q)\right) \text{ for } q \in [0,p],\ f \in [0,2p]$$



Q1. Find the curl of $x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$. What does the result say about the vector field?

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & z^2 \end{vmatrix} = [0]\hat{\mathbf{i}} - [0]\hat{\mathbf{j}} + [0]\hat{\mathbf{k}} = \mathbf{0}$$

Since the curl is the zero vector, we know that the vector field is conservative. This means that the vector field is actually the gradient of some scalar potential function. We can find this scalar potential function systematically:

$$\mathbf{F} = \nabla f$$

$$x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}} = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

$$\frac{\partial f}{\partial x} = x^2; \quad f(x,y,z) = \frac{1}{3}x^3 + g(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = y^2; \quad g(y,z) = \frac{1}{3}y^3 + h(z); \quad f(x,y,z) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + h(z)$$

$$\frac{\partial f}{\partial z} = \frac{dh}{dz} = z^2; \quad h(z) = \frac{1}{3}z^3 + C;$$

Thus, the scalar potential function is simply:

$$f(x,y,z) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 + C$$

Q2. What are the distance and unit direction vector between the points $(3, 0, 4)_{cylindrical}$ and $(5, p/2, 3p/2)_{spherical}$?

Convert all vectors to Cartesian coordinates first!

$$(r, f, z) = (3, 0, 4) \Rightarrow (3, 0, 4)_{cart}$$
$$x = r\cos(f) = 3\cos(0)$$
$$y = r\sin(f) = 3\sin(0)$$
$$z = 4$$

$$(r, q, f) = (5, p/2, 3p/2) \Rightarrow (0, -5, 0)_{cart}$$
$$x = r\cos(f)\sin(q) = 5\cos(3p/2)\sin(p/2) = 0$$
$$y = r\sin(f)\sin(q) = 5\sin(3p/2)\sin(p/2) = -5$$
$$z = r\cos(f) = 5\cos(3p/2) = 0$$

Distance is the norm of the difference between the two Cartesian vectors:

$$(3,0,4)_{cart} - (0,-5,0)_{cart} = (3,5,4)$$
$$\boxed{\left\|(3,5,4)\right\| = \sqrt{9+25+16} = \sqrt{50}}$$

Unit direction vector is simply the difference vector divided by its length:

$$\boxed{\frac{(3,5,4)}{\sqrt{50}}}$$