Q1. Give a brief physical interpretation for each of Maxwell's Four Equations and the Continuity Equation:

1. Gauss' Law for the Electric Field:

$$\oint_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}} = \int_{V} \rho_v dV \qquad \nabla \bullet \bar{\mathbf{D}} = \rho_v$$

Physical Meaning:

Integral Form:

The total ***outward flux*** *of the electric field density vector emanating from a closed surface (* $\oint_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}}$ *) is equal to the total charge distribution in the volume enclosed by the surface (* $\int_{V} \rho_v dV$ *).*

Point form:

The total electric field density vector emanating from a point ($\nabla \bullet \bar{\mathbf{D}}$ *) is equal to the total charge at that point (* ρ_v *).*

Note: Charges serve as sources for Electric Fields.

2. Gauss' Law for the Magnetic Field:

$$\oint_{\mathbf{s}} \bar{\mathbf{B}} \bullet d\bar{\mathbf{s}} = 0 \qquad\qquad \nabla \bullet \bar{\mathbf{B}} = 0$$

Physical Meaning:

Integral Form:

*The total **outward flux** of the magnetic field density vector emanating from a closed surface ($\oint_{\mathbf{s}} \bar{\mathbf{B}} \bullet d\bar{\mathbf{s}}$) is equal to zero ($0$) (i.e. no magnetic charges exist; magnetic field lines close in on themselves).*

Point form:

The total magnetic field density vector emanating from a point ($\nabla \bullet \bar{\mathbf{B}}$) is equal to the total magnetic charge at that point (0).

3. Faraday's Law

$$\oint_{\mathbf{c}} \bar{\mathbf{E}} \bullet d\bar{\mathbf{l}} = -\frac{d}{dt}\int_{\mathbf{s}} \bar{\mathbf{B}} \bullet d\bar{\mathbf{s}} \qquad \nabla \times \bar{\mathbf{E}} = -\partial_t \bar{\mathbf{B}}$$

Physical Meaning:

Integral Form:

The total ***circulation*** *of the electric field intensity vector around a contour C (* $\oint_{\mathbf{c}} \bar{\mathbf{E}} \bullet d\bar{\mathbf{l}}$ *) is equal to the time rate of decrease of the magnetic flux crossing the surface area bounded by the contour C (* $-\frac{d}{dt}\int_{\mathbf{s}} \bar{\mathbf{B}} \bullet d\bar{\mathbf{s}}$ *).*

Point form:

The ***circulation*** *of the electric field intensity vector at a point (* $\nabla \times \bar{\mathbf{E}}$ *) is equal to the time rate of decrease of the magnetic field density at that point (* $-\partial_t \bar{\mathbf{B}}$ *).*

Note: A time-varying Magnetic Field can produce an Electric Field.

4. Ampere-Maxwell Law

$$\oint_{\mathbf{c}} \bar{\mathbf{H}} \bullet d\bar{\mathbf{l}} = \int_{\mathbf{s}} \bar{\mathbf{J}} \bullet d\bar{\mathbf{s}} + \frac{d}{dt}\int_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}} \qquad \nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}} + \partial_t \bar{\mathbf{D}}$$

Physical Meaning:

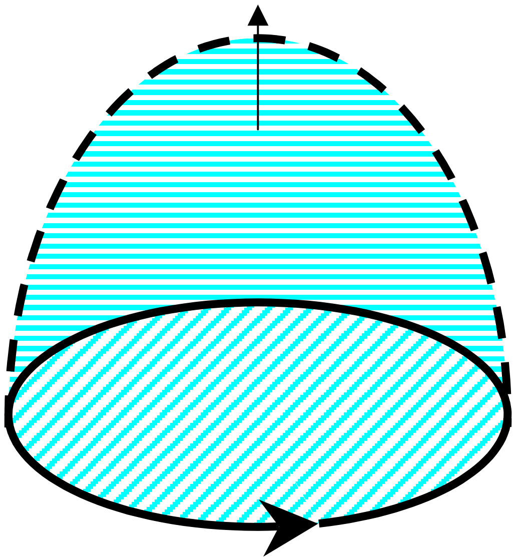
Integral Form:

The total ***circulation*** *of the magnetic field intensity vector around a contour C (*$\oint_{\mathbf{c}} \bar{\mathbf{H}} \bullet d\bar{\mathbf{l}}$*) is equal to the total current crossing the surface area bounded by the contour C (*$\int_{\mathbf{s}} \bar{\mathbf{J}} \bullet d\bar{\mathbf{s}} + \frac{d}{dt}\int_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}}$*).*

Point form:

The ***circulation*** *of the magnetic field intensity vector at a point (*$\nabla \times \bar{\mathbf{H}}$*) is equal to the total current at that point (*$\bar{\mathbf{J}} + \partial_t \bar{\mathbf{D}}$*).*

Notes: A time varying Electric Field can produce a Magnetic Field.
Currents serve as sources for Magnetic Fields.



5. The Continuity Equation (Conservation of Charge):

$$\oint_{\mathbf{s}} \bar{\mathbf{J}} \bullet d\bar{\mathbf{s}} = -\frac{d}{dt}\int_{\mathrm{v}} \rho_{\mathrm{v}} dV \qquad \nabla \bullet \bar{\mathbf{J}} = -\partial_{\mathrm{t}} \rho_{\mathrm{v}}$$

*The total **outward flux** of the current density vector emanating from a closed surface (* $\oint_{\mathbf{s}} \bar{\mathbf{J}} \bullet d\bar{\mathbf{s}}$ *) is equal to time rate of decrease of the charge in the volume enclosed by the surface (* $-\frac{d}{dt}\int_{\mathrm{v}} \rho_{\mathrm{v}} dV$ *).*

Point form:

The total current density vector emanating from a point ($\nabla \bullet \bar{\mathbf{J}}$ *) is equal to the net rate of decrease of the charge at that point (* $-\partial_{\mathrm{t}} \rho_{\mathrm{v}}$ *).*

Note: This equation is just another way to express the concept of the conservation of charge.

Q2. Determine the energy stored in the field for the following spherical charge density:

$$\rho = \begin{cases} r\rho_0 & r < a \\ 0 & a < r \end{cases}$$

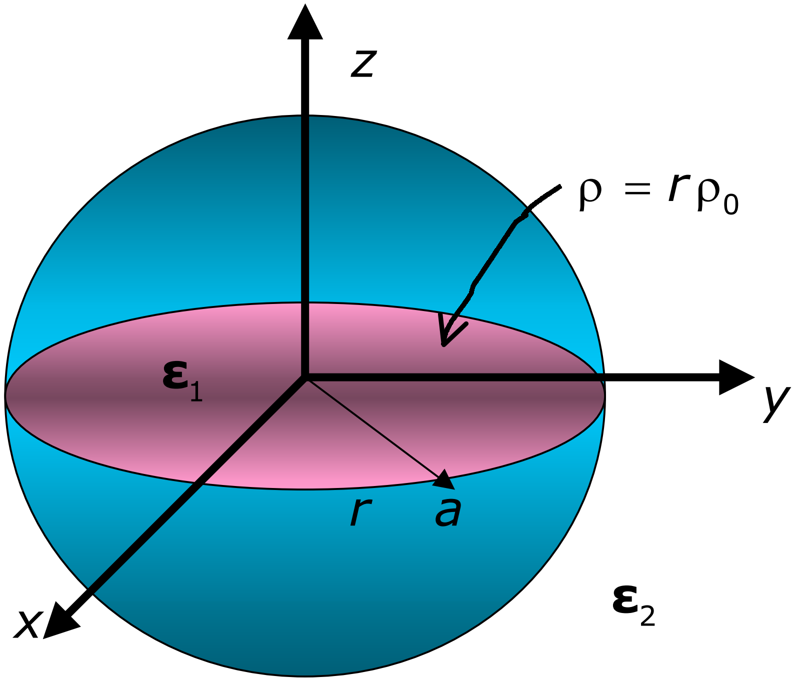
We are given a distribution of charge within a spherical volume, so we need to use spherical coordinates/vectors. We are asked to find the energy stored in the field produced by this source of charge. What kind of field is sourced by charge? Electric Fields! Thus, we need to use the Stored Energy equation for the Electric Field. To find the electric field, we need to use Gauss' Law for the Electric Field.

Equations we need:

$$W = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^{-1}\mathbf{D}^2 dV$$

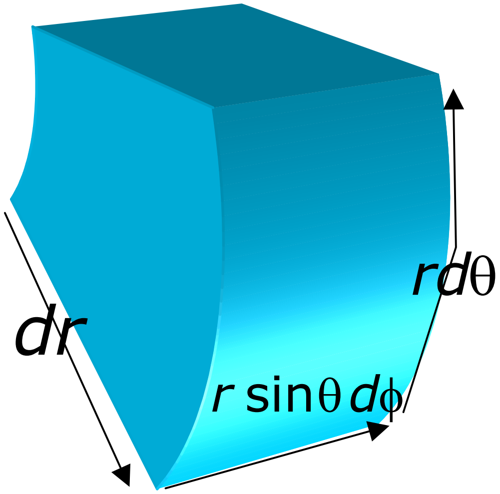
$$\oint_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}} = \int_v \rho_v dV$$

First, always draw a picture of the charge distribution:



Now, use Gauss' Law to find **D**: $\oint_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}} = \int_v \rho_v dV$

What is $d\mathbf{s}$ and dV? Look at a slice of the sphere:



$$d\mathbf{s} = \hat{\mathbf{n}} dA = r^2 \sin\theta\, d\phi\, d\theta\, \hat{\mathbf{r}}$$
$$dV = \left(r^2 \sin\theta\right) dr\, d\theta\, d\phi$$

$$\int_v \rho_v dV = \begin{cases} \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{r} \left(r\rho_0\right)\left(r^2 \sin\theta\right) dr\, d\theta\, d\phi = \rho_0 \int\limits_0^{2\pi} d\phi \int\limits_0^{\pi} \sin\theta\, d\theta \int\limits_0^{r} r^3 dr \Rightarrow \rho_0 \left(2\pi\right)\left(2\right)\left[\dfrac{r^4}{4}\right]_0^r \Rightarrow \pi\rho_0 r^4 & r < a \\ \int\limits_{\phi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{a} \left(r\rho_0\right)\left(r^2 \sin\theta\right) dr\, d\theta\, d\phi = \rho_0 \int\limits_0^{2\pi} d\phi \int\limits_0^{\pi} \sin\theta\, d\theta \int\limits_0^{a} r^3 dr \Rightarrow \rho_0 \left(2\pi\right)\left(2\right)\left[\dfrac{r^4}{4}\right]_0^a \Rightarrow \pi\rho_0 a^4 & r > a \end{cases}$$

$$\oint_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}} = \int\limits_0^{\pi} \int\limits_0^{2\pi} \bar{D}_r r^2 \sin\theta\, d\phi\, d\theta \Rightarrow 4\pi r^2 D_r$$

$$4\pi r^2 D_r = \begin{cases} \pi\rho_0 r^4 & r < a \\ \pi\rho_0 a^4 & r > a \end{cases}$$

$$D_r = \begin{cases} \dfrac{\rho_0 r^2}{4} & r < a \\ \dfrac{\rho_0 a^4}{4r^2} & r > a \end{cases}, \text{ thus, } \quad D_r^2 = \begin{cases} \dfrac{\rho_0^2 r^4}{16} & r < a \\ \dfrac{\rho_0^2 a^8}{16r^4} & r > a \end{cases}$$

Plugging this into the stored energy equation, we get:

$$\begin{aligned} W &= \frac{1}{2}\int_V \boldsymbol{\varepsilon}^{-1} D^2 \left(r^2 \sin\theta\right) dr\,d\theta\,d\phi \\ &= \frac{1}{2}\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_{r=0}^{\infty} \boldsymbol{\varepsilon}^{-1} r^2 D^2 dr \\ &= \frac{1}{2}\left[2\pi\right]\left[2\right]\left[\int_{r=0}^{r=a} \boldsymbol{\varepsilon}_1^{-1} r^2 \left(\frac{\rho_0^2 r^4}{16}\right) dr + \int_{r=a}^{r=\infty} \boldsymbol{\varepsilon}_2^{-1} r^2 \left(\frac{\rho_0^2 a^8}{16 r^4}\right) dr\right] \\ &= 2\pi\left[\left(\frac{\boldsymbol{\varepsilon}_1^{-1}\rho_0^2}{16}\right)\int_0^a r^6 dr + \left(\frac{\boldsymbol{\varepsilon}_2^{-1}\rho_0^2 a^8}{16}\right)\int_a^\infty \frac{1}{r^2} dr\right] \\ &= \left(\frac{\pi \boldsymbol{\varepsilon}_1^{-1}\rho_0^2}{8}\right)\left[\frac{r^7}{7}\right]_0^a + \left(\frac{\pi \boldsymbol{\varepsilon}_2^{-1}\rho_0^2 a^8}{8}\right)\left[\frac{-1}{r}\right]_a^\infty \end{aligned}$$

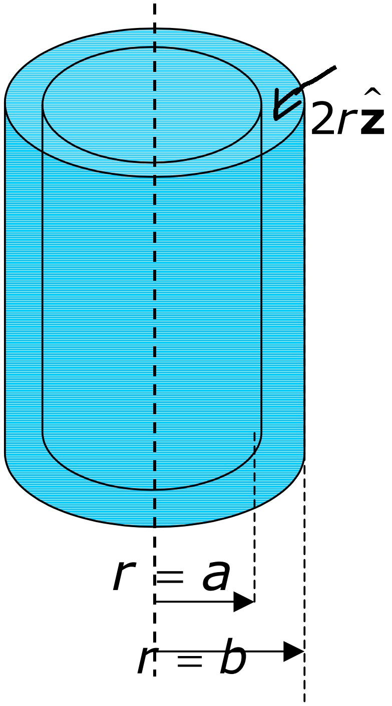
$$\boxed{W = \frac{\pi \boldsymbol{\varepsilon}_1^{-1}\rho_0^2 a^7}{56} + \frac{\pi \boldsymbol{\varepsilon}_2^{-1}\rho_0^2 a^7}{8}}$$

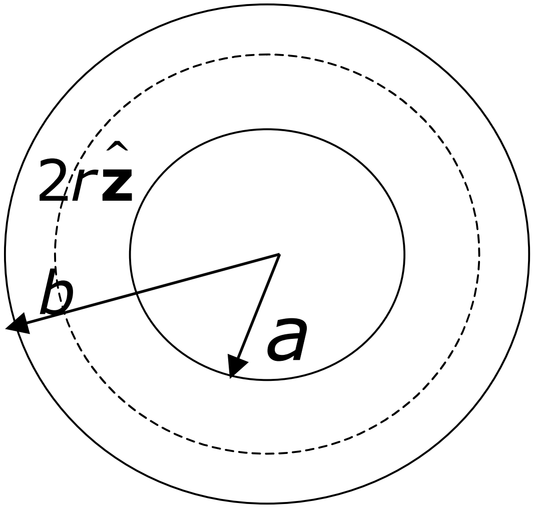
Q3. Determine the $\bar{\mathbf{H}}$ field for a current $\bar{\mathbf{J}} = \begin{cases} 2r\hat{\mathbf{z}} & a \le r \le b \\ 0 & other \end{cases}$, assuming that $\partial_t \bar{\mathbf{D}} = \mathbf{0}$.

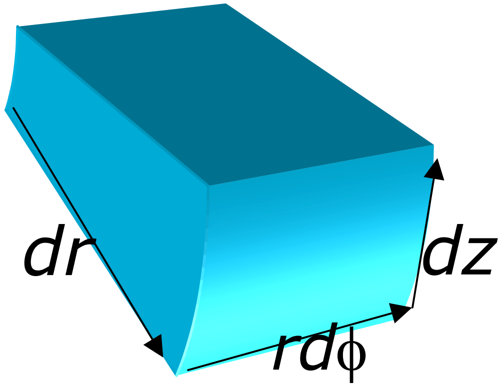
We are given a distribution of current within a cylindrical volume, so we need to use cylindrical coordinates/vectors. We are asked to find the magnetic field produced by this current source. Thus, we need to use the Ampere's Law. Due to the simple geometry, we use the integral form of Ampere's Law. DO NOT USE THE POINT FORM, it makes things more complicated than it should be.

Equations we need: $\oint_{\mathbf{c}} \bar{\mathbf{H}} \bullet d\bar{\mathbf{l}} = \int_{\mathbf{s}} \bar{\mathbf{J}} \bullet d\bar{\mathbf{s}} + \cancel{\frac{d}{dt}\int_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}}}$

First, draw a picture of the current distribution, and a slice of the cylinder:







Since $\bar{\mathbf{J}}$ is only in the $\hat{\mathbf{z}}$ direction, then by the RHR, $\bar{\mathbf{H}}$ becomes $\bar{\mathbf{H}} = H_\phi \widehat{\boldsymbol{\varphi}}$. What is $d\mathbf{l}$? Since $\bar{\mathbf{H}} = H_\phi \widehat{\boldsymbol{\varphi}}$, $d\mathbf{l}$ is the differential length in the $\widehat{\boldsymbol{\varphi}}$ direction, so $d\mathbf{l} = rd\phi\widehat{\boldsymbol{\varphi}}$. What is $d\mathbf{s}$? Since $\bar{\mathbf{J}}$ is in the $\hat{\mathbf{z}}$ direction, $d\mathbf{s}$ is the differential area in the $\hat{\mathbf{z}}$ direction, so $d\mathbf{s} = rdrd\phi\hat{\mathbf{z}}$.

$$\oint_{\mathbf{c}} \bar{\mathbf{H}} \bullet d\bar{\mathbf{l}} = \int_{\phi=0}^{2\pi} H_\phi r d\phi = 2\pi r H_\phi$$

$$\int_{\mathbf{s}} \bar{\mathbf{J}} \bullet d\bar{\mathbf{s}} = \begin{cases} 0 & 0 < R < a \\ \int_0^{2\pi}\int_a^R 2r^2 drd\phi \Rightarrow 4\pi\left[\frac{r^3}{3}\right]_a^R = \frac{4\pi}{3}\left[R^3 - a^3\right] & a < R < b \\ \int_0^{2\pi}\int_a^b 2r^2 drd\phi \Rightarrow 4\pi\left[\frac{r^3}{3}\right]_a^b = \frac{4\pi}{3}\left[b^3 - a^3\right] & R > b \end{cases}$$

$$2\pi r H_\phi = \begin{cases} 0 & 0 < R < a \\ \frac{4\pi}{3}\left[R^3 - a^3\right] & a < R < b \\ \frac{4\pi}{3}\left[b^3 - a^3\right] & R > b \end{cases}$$

$$\boxed{H_\phi = \begin{cases} 0 & 0 < R < a \\ \frac{2}{3}\frac{\left[R^3 - a^3\right]}{R} & a < R < b \\ \frac{2}{3}\frac{\left[b^3 - a^3\right]}{R} & R > b \end{cases}}$$

Q4. Given $\overline{\mathbf{V}} = x^2\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + \frac{1}{2}y^2\hat{\mathbf{z}}$ and $\partial_t\overline{\mathbf{D}} = \overline{\mathbf{J}} = 0$, can $\overline{\mathbf{V}}$ be a magnetic field?

We are given an arbitrary vector field, and asked to see if it is possible for the field to be a magnetic field. There are TWO constraining equations to use to test to see if this is a magnetic field, namely, Gauss' Law for the Magnetic Field and the Ampere-Maxwell Law. In equations, we are asking if this condition holds:

$$\overline{\mathbf{V}} = \overline{\mathbf{H}} = \frac{\overline{\mathbf{B}}}{\boldsymbol{\mu}}?$$

From Ampere-Maxwell,

$$\nabla \times \overline{\mathbf{H}} = \cancel{\overline{\mathbf{J}} + \partial_t\overline{\mathbf{D}}}$$

$$\nabla \times \overline{\mathbf{V}} = \mathbf{0}$$

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x^2 & yz & y^2 \end{vmatrix} = \mathbf{0}\text{, so } \overline{\mathbf{V}} \text{ satisfies the Ampere-Maxwell Law.}$$

From Gauss' Law for the Magnetic Field,

$$\nabla \bullet \overline{\mathbf{B}} = 0$$

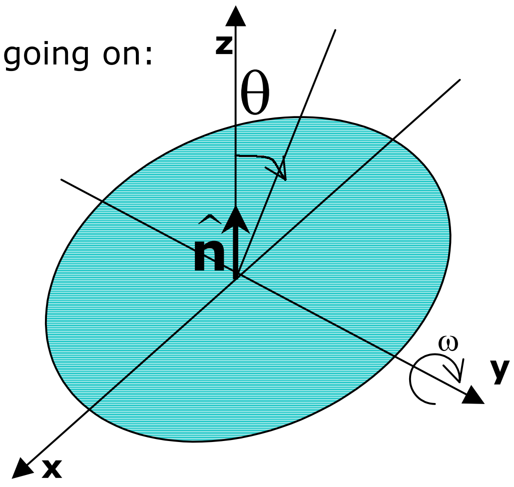
$$\nabla \bullet \overline{\mathbf{V}} = 0$$

$$\partial_x\left(x^2\right) + \partial_y\left(yz\right) + \partial_z\left(\frac{1}{2}y^2\right) = 2x + z \neq 0$$

Thus, since $\overline{\mathbf{V}}$ does not satisfy Maxwell's Equations, it cannot be a magnetic field.

Q5. A planar loop of area A, which is centered at the origin, is rotating around the y-axis at an angular frequency of ω. Find the emf if there is an arbitrary time varying magnetic field, $\overline{\mathbf{B}} = B_x(t)\hat{\mathbf{x}} + B_y(t)\hat{\mathbf{y}} + B_z(t)\hat{\mathbf{z}}$.

First, draw a picture to get an idea of what is going on:



Use Faraday's Law to calculate the EMF: $\varphi_{EMF} = -\dfrac{d}{dt}\int_{\mathbf{s}} \overline{\mathbf{B}} \bullet d\overline{\mathbf{s}}$.

What is $d\overline{\mathbf{s}}$? Well, $d\overline{\mathbf{s}} = \hat{\mathbf{n}}dA$, but here $\hat{\mathbf{n}}$ changes with angle $\theta = \omega t$, so we must find a suitable $\hat{\mathbf{n}}$:

$$\left.\begin{array}{c|c} \theta & \hat{\mathbf{n}} \\ \hline 0 & \hat{\mathbf{z}} \\ \hline \dfrac{\pi}{2} & -\hat{\mathbf{x}} \\ \hline \pi & -\hat{\mathbf{z}} \\ \hline \dfrac{3\pi}{2} & \hat{\mathbf{x}} \end{array}\right\} \Rightarrow \hat{\mathbf{n}} = \cos(\theta)\hat{\mathbf{z}} - \sin(\theta)\hat{\mathbf{x}} \Rightarrow \hat{\mathbf{n}} = \cos(\omega t)\hat{\mathbf{z}} - \sin(\omega t)\hat{\mathbf{x}}$$

Thus, $d\overline{\mathbf{s}} = \left[\cos(\omega t)\hat{\mathbf{z}} - \sin(\omega t)\hat{\mathbf{x}}\right]dA$.

$$\overline{\mathbf{B}} = \left(B_x(t), B_y(t), B_z(t)\right) \text{ and } d\overline{\mathbf{s}} = \left(-\sin(\omega t)\,dA, 0, \cos(\omega t)\,dA\right)\text{, so}$$

$$\overline{\mathbf{B}} \bullet d\overline{\mathbf{s}} = -B_x(t)\sin(\omega t)\,dA + B_z(t)\cos(\omega t)\,dA$$

$$\begin{aligned}
-\frac{d}{dt}\int_{\mathbf{s}} \overline{\mathbf{B}} \bullet d\overline{\mathbf{s}} &= \frac{d}{dt}\int_{\mathbf{s}} \left[B_x(t)\sin(\omega t) - B_z(t)\cos(\omega t)\right] dA \\
&= \frac{d}{dt}\left[B_x(t)\sin(\omega t) - B_z(t)\cos(\omega t)\right]\int_{\mathbf{s}} dA \\
&= A\frac{d}{dt}\left[B_x(t)\sin(\omega t) - B_z(t)\cos(\omega t)\right] \\
&= A\left[\frac{d}{dt}\left[B_x(t)\sin(\omega t)\right] - \frac{d}{dt}\left[B_z(t)\cos(\omega t)\right]\right] \\
&= A\left[\left[\omega B_x(t)\cos(\omega t) + \dot{B}_x(t)\sin(\omega t)\right] - \left[-\omega B_z(t)\sin(\omega t) + \dot{B}_z(t)\cos(\omega t)\right]\right]
\end{aligned}$$

$$\boxed{\varphi_{EMF} = \omega A\left[B_x(t)\cos(\omega t) + B_z(t)\sin(\omega t)\right] + A\left[\dot{B}_x(t)\sin(\omega t) - \dot{B}_z(t)\cos(\omega t)\right]}$$

Review of the Boundary Conditions (*fields at the interface between 2 media*) be able to prove these results and apply them to problems:

1. $\hat{\mathbf{n}}_{2\to1} \bullet \left(\overline{\mathbf{D}}_1 - \overline{\mathbf{D}}_2\right) = \rho_s$

   The **normal** components of $\overline{\mathbf{D}}$ ($\hat{\mathbf{n}}_{2\to1} \bullet \left(\overline{\mathbf{D}}_1 - \overline{\mathbf{D}}_2\right)$) are discontinuous by surface charge layer (ρ_s) at the interface between two media.

2. $\hat{\mathbf{n}}_{2\to1} \times \left(\overline{\mathbf{E}}_1 - \overline{\mathbf{E}}_2\right) = 0$

   The **tangential** components of $\overline{\mathbf{E}}$ ($\hat{\mathbf{n}}_{2\to1} \times \left(\overline{\mathbf{E}}_1 - \overline{\mathbf{E}}_2\right)$) are **always** continuous at the interface between two media.

3. $\hat{\mathbf{n}}_{2\to1} \bullet \left(\overline{\mathbf{B}}_1 - \overline{\mathbf{B}}_2\right) = 0$

   The **normal** components of $\overline{\mathbf{B}}$ ($\hat{\mathbf{n}}_{2\to1} \bullet \left(\overline{\mathbf{B}}_1 - \overline{\mathbf{B}}_2\right)$) are **always** continuous at the interface between two media.

4. $\hat{\mathbf{n}}_{2\to1} \times \left(\overline{\mathbf{H}}_1 - \overline{\mathbf{H}}_2\right) = \overline{\mathbf{J}}_S$

   The **tangential** components of $\overline{\mathbf{H}}$ ($\hat{\mathbf{n}}_{2\to1} \times \left(\overline{\mathbf{H}}_1 - \overline{\mathbf{H}}_2\right)$) are discontinuous by surface current density ($\overline{\mathbf{J}}_S$) at the interface between two media.

Notes:

1. $\hat{\mathbf{n}}_{2\to1}$ (normal to the surface pointing from medium 2 into medium 1).

2. Remember that $\overline{\mathbf{D}}_i = \boldsymbol{\varepsilon}_i \overline{\mathbf{E}}_i$ and $\overline{\mathbf{B}}_i = \boldsymbol{\mu}_i \overline{\mathbf{H}}_i$ where $i \in \{1,2\}$.

3. $\hat{\mathbf{n}}_{2\to1} \bullet$ selects the normal components (dot product).

4. $\hat{\mathbf{n}}_{2\to1} \times$ selects the tangential components (cross product).

Q6. Show that the energy stored in the electric field is $W = \frac{1}{2}\int_{\text{Space}} \bar{\mathbf{D}}\cdot\bar{\mathbf{E}}\,dV$.

Begin by bringing in charges into an empty space and calculate the work required to assemble the charges:

$$W = 0 + q_2\left(\phi_{1\to 2}\right) + q_3\left(\phi_{1\to 3} + \phi_{2\to 3}\right) + q_4\left(\phi_{1\to 4} + \phi_{2\to 4} + \phi_{3\to 4}\right),$$

where $\phi_{i\to j}$ is the potential from the ith charge on the jth charge. Now we can assemble the charges in reverse order and calculate the work needed to assemble the charges:

$$W = 0 + q_3\left(\phi_{4\to 3}\right) + q_2\left(\phi_{4\to 2} + \phi_{3\to 2}\right) + q_1\left(\phi_{4\to 1} + \phi_{3\to 1} + \phi_{2\to 1}\right).$$

Note that it takes the same amount of work to assemble the charges, so if we add up the two works, we get:

$$2W = q_1\phi_1 + q_2\phi_2 + q_3\phi_3 + q_4\phi_4,$$

where ϕ_i is the "local" potential, i.e., the total potential experienced by the ith charge. We can generalize this equation as a sum of a discrete set of charges:

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i\phi_i .$$

Now for a continuous charge distribution, we can move the finite sum to a volume integral:

$$W = \frac{1}{2}\int_V \rho\phi\, dV .$$

From Gauss' Law, we know that $\nabla \bullet \bar{\mathbf{D}} = \rho_v$, so substituting this into the integral, we obtain,

$$W = \frac{1}{2}\int_V \phi\nabla \bullet \bar{\mathbf{D}}\, dV .$$

From Vector Calculus, we know that $\nabla \bullet \left(\phi\bar{\mathbf{D}}\right) = \nabla\phi \bullet \bar{\mathbf{D}} + \phi\nabla \bullet \bar{\mathbf{D}}$, so that $\phi\nabla \bullet \bar{\mathbf{D}} = \nabla \bullet \left(\phi\bar{\mathbf{D}}\right) - \nabla\phi \bullet \bar{\mathbf{D}}$:

$$W = \frac{1}{2}\int_V \left[\nabla \bullet \left(\phi\bar{\mathbf{D}}\right) - \nabla\phi \bullet \bar{\mathbf{D}}\right] dV$$

Now, we use the Divergence Theorem to write the first volume integral as a surface integral:

$$W = \underbrace{\cancel{\frac{1}{2}\int_S \phi \bar{\mathbf{D}} \nabla \bullet d\mathbf{s}}}_{=0} - \frac{1}{2}\int_V \nabla\phi \bullet \bar{\mathbf{D}} dV$$

Examining the first term, this is a surface integral over all space, and we define the potential at infinity to be 0, or else we would have infinite energy. Thus,

$$W = -\frac{1}{2}\int_V \nabla\phi \bullet \bar{\mathbf{D}} dV \text{ , but } -\nabla\phi = \mathbf{E}$$

$$\boxed{W = \frac{1}{2}\int_V \mathbf{E} \bullet \bar{\mathbf{D}} dV}$$

Q7. Obtain the point form of Maxwell's Equations from the Integral Form.

$\oint_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}} = \int_{v} \rho_v dV$ Use the Divergence Theorem to convert the surface integral to a volume integral.

$\int_{V} \nabla \bullet \bar{\mathbf{D}} dV = \int_{v} \rho_v dV$

$\int_{V} \left[\nabla \bullet \bar{\mathbf{D}} - \rho_v \right] dV = 0$

$\nabla \bullet \bar{\mathbf{D}} = \rho_v$

$\oint_{\mathbf{s}} \bar{\mathbf{B}} \bullet d\bar{\mathbf{s}} = 0$ Use the Divergence Theorem to convert the surface integral to a volume integral.

$\int_{V} \nabla \bullet \bar{\mathbf{B}} dV = 0$

$\nabla \bullet \bar{\mathbf{B}} = 0$

$\oint_{\mathbf{c}} \bar{\mathbf{E}} \bullet d\bar{\mathbf{l}} = -\frac{d}{dt} \int_{\mathbf{s}} \bar{\mathbf{B}} \bullet d\bar{\mathbf{s}}$ Use Stoke's Theorem to convert the line integral to a surface integral

$\int_{S} \nabla \times \bar{\mathbf{E}} \bullet d\bar{\mathbf{s}} = -\frac{d}{dt} \int_{\mathbf{s}} \bar{\mathbf{B}} \bullet d\bar{\mathbf{s}}$

$\nabla \times \bar{\mathbf{E}}\left(x_0, y_0, z_0\right) = -\frac{d}{dt} B\left(x_0, y_0, z_0\right)$ Shrink the surface to a single point

$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\cancel{\frac{dx}{dt}} + \frac{\partial}{\partial y}\cancel{\frac{dy}{dt}} + \frac{\partial}{\partial z}\cancel{\frac{dz}{dt}}$ Remember the total derivative expansion

$\nabla \times \bar{\mathbf{E}} = -\partial_t \bar{\mathbf{B}}$

$\oint_{\mathbf{c}} \bar{\mathbf{H}} \bullet d\bar{\mathbf{l}} = \int_{\mathbf{s}} \bar{\mathbf{J}} \bullet d\bar{\mathbf{s}} + \frac{d}{dt}\int_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}}$ Use Stoke's Theorem to convert the line integral to a surface integral

$$\int_{S} \nabla \times \bar{\mathbf{H}} \bullet d\bar{\mathbf{s}} = \int_{\mathbf{s}} \bar{\mathbf{J}} \bullet d\bar{\mathbf{s}} + \frac{d}{dt}\int_{\mathbf{s}} \bar{\mathbf{D}} \bullet d\bar{\mathbf{s}}$$

$\nabla \times \bar{\mathbf{H}}\left(x_0, y_0, z_0\right) = \bar{\mathbf{J}}\left(x_0, y_0, z_0\right) + \frac{d}{dt}\bar{\mathbf{D}}\left(x_0, y_0, z_0\right)$ Shrink the surface to a single point

$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}\cancel{\frac{dx}{dt}} + \frac{\partial}{\partial y}\cancel{\frac{dy}{dt}} + \frac{\partial}{\partial z}\cancel{\frac{dz}{dt}}$ Remember the total derivative expansion

$$\nabla \times \bar{\mathbf{H}} = \bar{\mathbf{J}} + \partial_t \bar{\mathbf{D}}$$

Q8. Derive the Electromagnetic Wave Equation (both E and H):

To get the **E** wave equation, take the time derivative of Ampere's Law, $\partial_t\left[\nabla\times\bar{\mathbf{H}} = \bar{\mathbf{J}} + \partial_t\bar{\mathbf{D}}\right]\Big|_{\bar{\mathbf{J}}=\sigma\bar{\mathbf{E}}}$ and convert to an equation in **E:**

$\nabla\times\partial_t\bar{\mathbf{H}} = \sigma\partial_t\bar{\mathbf{E}} + \partial_t^2\bar{\mathbf{D}}$

$\nabla\times\left(\mu^{-1}\partial_t\bar{\mathbf{B}}\right) = \sigma\partial_t\bar{\mathbf{E}} + \varepsilon\partial_t^2\bar{\mathbf{E}}$ From Faraday's Law, $\partial_t\bar{\mathbf{B}} = -\nabla\times\bar{\mathbf{E}}$

$-\mu^{-1}\nabla\times\left(\nabla\times\bar{\mathbf{E}}\right) = \sigma\partial_t\bar{\mathbf{E}} + \varepsilon\partial_t^2\bar{\mathbf{E}}$ Recall that $\nabla\times\nabla\times\bar{\mathbf{A}} = \mathit{grad}\left(\mathit{div}\left(A\right)\right) - \mathit{laplacian}\left(A\right)$

$-\mu^{-1}\left[\nabla\left(\nabla\bullet\bar{\mathbf{E}}\right) - \nabla^2\bar{\mathbf{E}}\right] = \sigma\partial_t\bar{\mathbf{E}} + \varepsilon\partial_t^2\bar{\mathbf{E}}$ From Gauss' Law, $\nabla\bullet\bar{\mathbf{E}} = \frac{\rho}{\varepsilon}$

$-\mu^{-1}\left[\cancel{\nabla\left(\frac{\rho}{\varepsilon}\right)} - \nabla^2\bar{\mathbf{E}}\right] = \sigma\partial_t\bar{\mathbf{E}} + \varepsilon\partial_t^2\bar{\mathbf{E}}$ We take the location of the wave "far away" from its charge source.

$\boxed{\nabla^2\bar{\mathbf{E}} = \mu\sigma\partial_t\bar{\mathbf{E}} + \mu\varepsilon\partial_t^2\bar{\mathbf{E}}}$

To get the **H** wave equation, take the time derivative of Faraday's Law, and convert to an equation in **H**:

$\partial_t\left[\nabla\times\bar{\mathbf{E}} = -\partial_t\bar{\mathbf{B}}\right]$

$\nabla\times\partial_t\bar{\mathbf{E}} = -\partial_t^2\bar{\mathbf{B}}$ Use Ampere's Law to get $\nabla\times\bar{\mathbf{H}} = \sigma\bar{\mathbf{E}} + \partial_t\bar{\mathbf{D}} \Rightarrow \nabla\times\bar{\mathbf{H}} = \sigma\bar{\mathbf{E}} + \varepsilon\partial_t\bar{\mathbf{E}} \Rightarrow \partial_t\bar{\mathbf{E}} = \varepsilon^{-1}\left(\nabla\times\bar{\mathbf{H}}\right) - \varepsilon^{-1}\sigma\bar{\mathbf{E}}$

$\nabla\times\left[\varepsilon^{-1}\left(\nabla\times\bar{\mathbf{H}}\right) - \varepsilon^{-1}\sigma\bar{\mathbf{E}}\right] = -\mu\partial_t^2\bar{\mathbf{H}}$

$\varepsilon^{-1}\nabla\times\nabla\times\bar{\mathbf{H}} - \varepsilon^{-1}\sigma\left(\nabla\times\bar{\mathbf{E}}\right) = -\mu\partial_t^2\bar{\mathbf{H}}$ Use Faraday's Law $\nabla\times\bar{\mathbf{E}} = -\mu\partial_t\bar{\mathbf{H}}$

$\varepsilon^{-1}\left[\cancel{\nabla\left(\nabla\bullet\bar{\mathbf{H}}\right)} - \nabla^2\bar{\mathbf{H}}\right] + \varepsilon^{-1}\mu\sigma\partial_t\bar{\mathbf{H}} = -\mu\partial_t^2\bar{\mathbf{H}}$

$\boxed{\nabla^2\bar{\mathbf{H}} = \mu\sigma\partial_t\bar{\mathbf{H}} + \mu\varepsilon\partial_t^2\bar{\mathbf{H}}}$