ANALYSIS OF A SPACE CURVE

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This note goes through the analysis of a space curve with reference to the example

$$\mathbf{r}(t) = < \frac{1}{2}t^2, \frac{1}{3}(2t)^{3/2}, t > .$$

Here $\mathbf{r}$ is a position vector the the point specified by the parameter t. Note that the curve is defined only for $t \geq 0$. This parameter need not be time, but the curve acquires a physical interpretation by regarding it as being traced by a moving point whose position is given at time t.

Differentiating gives

$$\mathbf{r}'(t) = \; < \frac{1}{2} \cdot 2t, \frac{1}{3} \cdot \frac{3}{2}(2t)^{1/2} \cdot [2], 1 > \; = \; < t, \sqrt{2t}, 1 > .$$

If t is time, this is the velocity vector $\mathbf{v}$ of a moving point tracing the curve. Since the velocity is continuous and never $\mathbf{0}$, the curve is smooth everywhere.

The length of the derived vector is

$$||\mathbf{r}'|| = (t^2 + 2t + 1)^{1/2} = t + 1.$$

This non-negative scalar can be interpreted as the speed of the tracing point. The tracing point starts suddenly at $t = 0$ with positive speed $||\mathbf{v}(0)|| = 0 + 1 = 1$.

The *arc length* from $t = a$ to $t = b$ is the distance s along the curve from the first point to the second, positive in that direction. From the Pythagorean theorem $\Delta s \geq \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2 + (\frac{\Delta z}{\Delta t})^2}\ \Delta t$, where $\Delta t = b - a$. Partitioning the interval into many smaller subintervals and adding these calculations of the chord lengths gives a better approximation. As the length of the longest subinterval goes to zero, the error between the curve and the chords goes to zero also, and the approximation becomes a definite integral. This results in the formula

$$s = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\ dt = \int_a^b ||\mathbf{r}'(t)||\ dt.$$

Since $||\mathbf{r}'(t)||$ is the speed of the point along the curve, this agrees with our intuition of distance as the integral of speed. In applying this formula it is important that the tracing point not reverse direction (it cannot while the curve remains smooth) and that the curve be traced only once.

Calculating the arc length from $t = 0$ to $t = 4$ in our example gives

$$s = \int_0^4 ||\mathbf{r}'(t)||dt = \int_0^4 (t+1)dt = (\frac{t^2}{2} + t)|_0^4 = (\frac{4^2}{2} + 4) - 0 = 12.$$

This example was specially selected to provide a simple expression for speed, so that this integration would be reasonable to carry out.

The formula for arc length is clearly different for different parametric representations. Arc length is a geometrical property of the curve, however, and it seems most natural to describe position on the curve by a coordinate giving a directed length from a fixed origin. This is essentially finding a parametrization in which the tracing point moves with constant speed. This can be done by integrating to get the arc length to a variable point at parameter t and solving for t in terms of s, but the result is usually forbiddingly complicated. It is valuable conceptually but is seldom written down.

The velocity vector is tangent to the curve at the given point. If $\mathbf{v} \neq \mathbf{0}$ the physical quantity speed can be divided out, producing the unit tangent vector $\mathbf{T}$. In our example this gives

$$\mathbf{T} = \frac{\mathbf{r}'}{||\mathbf{r}'||} = < \frac{t}{t+1}, \frac{\sqrt{2t}}{t+1}, \frac{1}{t+1} > = \frac{< t, \sqrt{2t}, 1 >}{t+1}.$$

The unit tangent vector indicates only the geometrical information about the direction of the tangent line and the residual sense of direction retained from the parametrization. .

More information about the curve and how it evolves can be obtained from the second derivative

$$\mathbf{r}''(t) = \ < 1, \frac{1}{\sqrt{2t}}, 0 > .$$

If t is time, this is the acceleration vector $\mathbf{a}$ of the tracing point. Note that $\mathbf{a}(0)$ does not exist, consistent with the abrupt beginning of the motion.

The scalar projection of acceleration along the tangent is denoted a_T. This is easily calculated by

$$a_T \ = \mathbf{a} \bullet \mathbf{T} = \frac{\mathbf{a} \bullet \mathbf{v}}{||\mathbf{v}||}.$$

This is the effect of the acceleration in changing the speed only, without changing the direction. This can be verified by a direct calculation of the derivative of speed:

$$\frac{d}{dt}||\mathbf{v}|| = \frac{d}{dt}(\mathbf{v} \bullet \mathbf{v})^{1/2} = \frac{1}{2}(\mathbf{v} \bullet \mathbf{v})^{-1/2}\frac{d}{dt}(\mathbf{v} \bullet \mathbf{v}) = \frac{1}{2||\mathbf{v}||}(2\mathbf{v}' \bullet \mathbf{v}) = \mathbf{v}' \bullet \frac{\mathbf{v}}{||\mathbf{v}||} = \mathbf{a} \bullet \mathbf{T}.$$

In the example it is easy to calculate $s'(t) = \frac{d}{dt}(t+1) = 1$. The direct calculation of the scalar tangential projection gives

$$a_T = < 1, \frac{1}{\sqrt{2t}}, 0 > \bullet \frac{< t, \sqrt{2t}, 1 >}{t+1} = \frac{t+1}{t+1} = 1.$$

The tracing point moves along the curve with speed increasing at one unit per unit time. If this were the only effect of acceleration, the curve would be a straight line with the traced according to the same speed function $s(t) = t+1$.

The tangential vectorcomponent of acceleration is $\mathbf{PROJ}_{\mathbf{T}}(\mathbf{a}) = a_T\mathbf{T}$. In general this component of acceleration may have different length or even reversed direction relative to the velocity vector. In the example $a_T\mathbf{T} = 1 \cdot \mathbf{T} = \mathbf{T}$. This is a consequence of the choice of formulas in the example to give the simplest calculus.

A vector orthogonal to the tangent is said to be *normal* to the curve at the given point. There are infinitely many such vectors. Those lying in the plane of $\mathbf{v}$ and $\mathbf{a}$ and having positive projection along **a** are called *principal* normals. If **a** is parallel to **v**, no principal normal exists. Otherwise one can be constructed by finding the vector component of **a** orthogonal to **v**. Since $\mathbf{a} = \mathbf{PROJ_T}(\mathbf{a}) + \mathbf{ORTH_T}(\mathbf{a})$, we have .

$$\mathbf{ORTH_T}(\mathbf{a}) = \mathbf{a} - \mathbf{PROJ_T}(\mathbf{a}) = \mathbf{a} - a_T\mathbf{T}.$$

The component of acceleration orthogonal to **T** is the effect of the acceleration in changing the direction of the curve. If *all* the acceleration were in this component, the curve would be the same shape but the point would move at unit speed. In the example we calculate

$$\mathbf{ORTH_T}(\mathbf{a}) = \mathbf{a} - \mathbf{T} = \frac{< (t+1)\sqrt{2t}, t+1, 0 >}{(t+1)\sqrt{2t}} - \frac{< t\sqrt{2t}, 2t, \sqrt{2t} >}{(t+1)\sqrt{2t}} = \frac{< \sqrt{2t}, 1-t, -\sqrt{2t} >}{(t+1)\sqrt{2t}}.$$

If the length is divided out of a principal normal to remove the physical information contained in the acceleration, the result is the *principal unit normal vector* **N**. This length in the example is

$$||\mathbf{ORTH_T}(\mathbf{a})|| = \frac{|| < \sqrt{2t}, 1-t, -\sqrt{2t} > ||}{(t+1)\sqrt{2t}} = \frac{(2t+1-2t+t^2+2t)^{1/2}}{(t+1)\sqrt{2t}} = \frac{1}{\sqrt{2t}}.$$

Therefore in the example the principal unit normal vector is

$$\mathbf{N} = \frac{\mathbf{ORTH_T}(\mathbf{a})}{||\mathbf{ORTH_T}(\mathbf{a})||} = \frac{< \sqrt{2t}, 1-t, -\sqrt{2t} >}{(t+1)\sqrt{2t}} / \frac{1}{\sqrt{2t}} = \frac{< \sqrt{2t}, 1-t, -\sqrt{2t} >}{(t+1)}.$$

There is only one principal unit normal when it exists, and this is one of several ways to calculate it. This vector contains the geometrical information about the direction in which the curve is bending.

The scalar projection of acceleration normal to the curve is also the scalar projection of acceleration orthogonal to the tangent, and is denoted a_N. Thus $a_N = ||\mathbf{ORTH_T}(\mathbf{a})||$ and is always positive. Therefore $a_N\mathbf{N} = \mathbf{ORTH_T}(\mathbf{a})$ and $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$. Another way to calculate a_N is by the Pythagorean theorem:

$$a_N = \sqrt{||\mathbf{a}||^2 - (a_T)^2}$$

If θ is the angle from **v** to **a**, then $a_N = ||\mathbf{a}|| \sin\theta$. In three-dimensional space the relation of the cross-product to $\sin\theta$ gives yet another way to calculate a_N:

$$a_N = ||\mathbf{T} \times \mathbf{a}|| = \frac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||}.$$

This formula is especially convenient for numerical calculations, but for formulas other approaches may be no harder or even easier, depending on what has already been calculated.

The principal unit normal gives the direction in which the curve bends. But how sharply does it bend? A way to measure bending is to examine the change in the unit tangent vector. Sharper bending will produce a larger change of direction in the same distance. It is important to use arc length s rather than the "time" parameter t, since using t would re-introduce speed, which was carefully removed by use of the unit tangent. Thus to get information about the bending of the curve it is necessary study $\frac{d\mathbf{T}}{ds}$, the derivative of the unit tangent with respect to arc length. Since $\mathbf{T}$ always has unit length, $\frac{d\mathbf{T}}{ds}$ is always perpendicular to $\mathbf{T}$.

To avoid finding an arc length parametrization we use the chain rule, which gives $\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{d\mathbf{T}/dt}{||\mathbf{v}||}$. Now $\frac{d\mathbf{T}}{dt} = \frac{d}{dt}(\frac{1}{||\mathbf{v}||}\mathbf{v}) = (\frac{-1}{||\mathbf{v}||^2}\frac{d}{dt}||\mathbf{v}||)\mathbf{v} + \frac{1}{||\mathbf{v}||}\frac{d\mathbf{v}}{dt}$. Now $\frac{d}{dt}||\mathbf{v}|| = \mathbf{a} \bullet \frac{\mathbf{v}}{||\mathbf{v}||}$, so that $\frac{-1}{||\mathbf{v}||^2}\frac{d}{dt}||\mathbf{v}|| = \frac{-1}{||\mathbf{v}||^2}(\mathbf{a} \bullet \frac{\mathbf{v}}{||\mathbf{v}||}) = -\frac{\mathbf{a} \bullet \mathbf{v}}{||\mathbf{v}||^3}$. Thus $\frac{d\mathbf{T}}{dt} = -\frac{\mathbf{a} \bullet \mathbf{v}}{||\mathbf{v}||^3}\mathbf{v} + \frac{1}{||\mathbf{v}||}\mathbf{a} = \frac{1}{||\mathbf{v}||}(\mathbf{a} - (\mathbf{a} \bullet \frac{\mathbf{v}}{||\mathbf{v}||})\frac{\mathbf{v}}{||\mathbf{v}||}) = \frac{\mathbf{a} - a_T\mathbf{T}}{||\mathbf{v}||} = \frac{1}{||\mathbf{v}||}\mathbf{ORTH_T}(\mathbf{a})$. Then $\frac{d\mathbf{T}}{ds} = \frac{1}{||\mathbf{v}||^2}\mathbf{ORTH_T}(\mathbf{a})$. Thus both of these derivatives are principal normals, and dividing out their lengths produces $\mathbf{N}$. The length $||\frac{d\mathbf{T}}{ds}||$ is a geometrically meaningful measure of the sharpness of bending, called the curvature K.

Calculating $\frac{d\mathbf{T}}{ds}$ is often difficult, and using the relation derived above is frequently easier. The various expressions for a_N give other convenient ways to calculate K. Thus we have

$$\frac{d\mathbf{T}}{ds} = \frac{1}{||\mathbf{v}||^2}\mathbf{ORTH_T}(\mathbf{a}) = K\mathbf{N}$$

and

$$K = \frac{a_N}{||\mathbf{v}||^2} = \frac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||^3} = \frac{||\mathbf{T} \times \mathbf{a}||}{||\mathbf{v}||^2} = \frac{\sqrt{||\mathbf{a}||^2 - a_N^2}}{||\mathbf{v}||^2}.$$

In our example, we have $K = \frac{a_N}{||\mathbf{v}||^2} = (\frac{1}{\sqrt{2t}})/(t+1)^2 = \frac{1}{\sqrt{2t}(t+1)^2}$.

For a curve in 3-space the unit tangent and unit principal normal determine its orientation at all times. The third dimension in this moving coordinate system is the *binormal* $\mathbf{B}$, defined as the cross product of $\mathbf{T}$ and $\mathbf{N}$ in that order. Thus $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Differentiating the binormal with respect to arc length gives the direction in which the curve turns out of its plane and twists into a three-dimensional form. A measure of this rate of twisting, analogous to curvature, is obtained by taking the length of this derivative, $||\frac{d\mathbf{B}}{ds}||$. This is called the *torsion* of the curve. We will not explore the binormal or the torsion in this course.