Chapter 2 : Linear Equations
Practical Example

Q : An airplane travels between 2 cities 5000 km apart. The trip one way against the wind takes 6.25 hours while the return trip same day in direction of wind takes 5 hours. Find the speed of the airplane and wind

A : Let \( x \) be speed of airplane & \( y \) wind speed

\[
\begin{align*}
6.25(x-y) &= 5000 \\
5 \ (x+y) &= 5000
\end{align*}
\]

\( x= 900 \text{ km/hr} \ ; \ y=100 \text{ km/hr} \)
System of Linear Equations

Linear Equations establish connection between Algebra and Matrices

\[
\begin{align*}
\begin{array}{cccc}
{a}_{11}x_1 & + & {a}_{12}x_2 & + & \cdots & + & {a}_{1n}x_n & = b_1 \\
{a}_{21}x_1 & + & {a}_{22}x_2 & + & \cdots & + & {a}_{2n}x_n & = b_2 \\
& & & & \vdots & & & \vdots \\
{a}_{m1}x_1 & + & {a}_{m2}x_2 & + & \cdots & + & {a}_{mn}x_n & = b_m
\end{array}
\end{align*}
\]

We have “m” Linear Equations in “n” unknowns

Left hand side is a linear combination of column vectors. To derive the matrix representation, we need to first learn how to multiply matrices and vectors
**What is a Matrix?**

**Definition:** An m x n matrix A is an array of “m” rows and “n” columns of the form

\[
A = [a_{ij}] = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

\(a_{ij} \in \mathbb{R}\) (real number)

If m=n, A is called a square matrix. Otherwise, it is a rectangular matrix.

\[
= \begin{bmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_n
\end{bmatrix} = \begin{bmatrix}
    a_1^T \\
    a_2^T \\
    \vdots \\
    a_m^T
\end{bmatrix}
\]

“m” rows

“n” columns
Example

Average temperature in Dallas for the 12 months of the year over 10 years

<table>
<thead>
<tr>
<th></th>
<th>04</th>
<th>03</th>
<th></th>
<th>95</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>43</td>
<td>46</td>
<td>------</td>
<td>51</td>
</tr>
<tr>
<td>Feb</td>
<td>50</td>
<td>49</td>
<td></td>
<td>53</td>
</tr>
<tr>
<td>Dec</td>
<td>53</td>
<td>47</td>
<td>------</td>
<td>48</td>
</tr>
</tbody>
</table>

2-dimensional array = matrix

Can also consider different hours of day or days of year and different locations (higher-dimensional arrays)
Scalar

- A quantity described by a single real number

  e.g. Intensity of each voxel (volumetric pixel) in an MRI scan
Vector

- A one-dimensional array of numbers

EXAMPLE:

VECTOR=

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_n
\end{bmatrix}
\]
i.e. a column of numbers
Matrix

- A two-dimensional array of numbers
- Can inform about intensity of full rectangular grid
- Vector is just a $n \times 1$ matrix
Matrices

Matrix size defined as rows x columns (R x C)

\[ d_{ij} : i^{th} \text{ row}, j^{th} \text{ column} \]

Square (3 x 3)

\[
A = \begin{bmatrix}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9 \\
\end{bmatrix}
\]

Rectangular (3 x 2)

\[
A = \begin{bmatrix}
1 & 4 \\
2 & 5 \\
3 & 6 \\
\end{bmatrix}
\]

3 dimensional array (3 x 3 x 5)
# Matrices in MATLAB

## Description

**Matrix(X)**

Type into MATLAB: `X=[1 4 7;2 5 8;3 6 9]`

Meaning: \( X = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \)

Note: `;` = end of a row

## Reference Matrix Values (\( X(\text{row},\text{column}) \))

Note the `:` refers to all of row or column and `,` is divider between rows and columns

<table>
<thead>
<tr>
<th>Description</th>
<th>Type into MATLAB</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>3rd row</td>
<td>( X(3,:) )</td>
<td>( [3 \ 6 \ 9] )</td>
</tr>
<tr>
<td>2nd Element of 3rd column</td>
<td>( X(2,3) )</td>
<td>( 8 )</td>
</tr>
<tr>
<td>Elements 1&amp;2 of column 2</td>
<td>( X([1 \ 2],2) )</td>
<td>( [4 \ 5] )</td>
</tr>
</tbody>
</table>

## Special Types of Matrix

- **All zeros size 3x1**
  
  Type into MATLAB: `zeros(3,1)`
  
  Meaning: \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \)

- **All ones size 2x2**
  
  Type into MATLAB: `ones(2,2)`
  
  Meaning: \( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \)
Operations on Matrices

1. Transpose: \[ B = A^T \]
   \[ [b_{ij}] = [a_{ji}] \]
   
   \( A \) is \( m \times n \)
   
   \( B \) is \( n \times m \)

   Rows of \( A \) become columns of \( B \) and vice versa.

2. Adding 2 matrices: \( C = A + B \)
   
   \( c_{ij} = a_{ij} + b_{ij} \)  
   element-wise addition

   \( A \) & \( B \) should be of same size
Operations on Matrices

3. Multiply a matrix by a vector: (dimensions must be compatible)

**Approach I**

\[ Ab_1 = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} b_1 = \begin{bmatrix} a_1^T b_1 \\ a_2^T b_1 \\ \vdots \\ a_m^T b_1 \end{bmatrix} \]

or

**Approach II**

\[ Ab_1 = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = a_1 b_{11} + a_2 b_{21} + \cdots + a_n b_{n1} \]

Inner product with rows of A

Divide and Conquer!
Scalar Multiplication

4. Scalar * matrix = scalar multiplication

\[ \lambda \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b & \lambda c \\ \lambda d & \lambda e & \lambda f \end{pmatrix} \]
5. Multiply Two Matrices

\[ C = A \begin{bmatrix} B \end{bmatrix} \]

\[ m \times l \quad m \times n \quad n \times l \]

\[ C_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq l \]

Example:

\[ AB = A \begin{bmatrix} b_1 & b_2 & \cdots & b_l \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_l \end{bmatrix} \]

\[
\begin{bmatrix}
  a_1^T \\
  a_2^T \\
  \vdots \\
  a_m^T
\end{bmatrix}
\begin{bmatrix}
  a_1^T B \\
  a_2^T B \\
  \vdots \\
  a_m^T B
\end{bmatrix}
\]

\[ \rightarrow \text{We know how to multiply a matrix by a vector!} \]

Example: Calculate

\[
\begin{bmatrix}
  1 & 2 & -1 \\
  2 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & -1 \\
  2 & 0 \\
  0 & 1
\end{bmatrix} = ?
\]
Matrix Multiplication

“When A is a \( m \times n \) matrix & B is a \( k \times l \) matrix, \( AB \) is only possible if \( n=k \). The result will be an \( m \times l \) matrix.”

Simply put, can ONLY perform A*B IF:

Number of columns in A = Number of rows in B

Hint:

If you see this message in MATLAB:

```
??? Error using ==> mtimes
Inner matrix dimensions must agree
```

-Then columns in A is not equal to rows in B
Matrix Multiplication

• Multiplication method:
  Sum over product of respective rows and columns

\[
AB = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23}
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22} \\
b_{31} & b_{32}
\end{pmatrix}
= \begin{pmatrix}
a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\
a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}
\end{pmatrix}
\]

2×3  3×2  2×2

MATLAB does all this for you!
Simply type: \texttt{C = A * B}

Hint:
You can work out the size of the output (2x2). In MATLAB, if you pre-allocate a matrix this size (e.g. \texttt{C=zeros(2,2)}) then the calculation is quicker
1. In general $AB \neq BA$ (non-commutative unlike scalars) $BA$ might not even be defined!

Example:

\[
\begin{bmatrix}
1 & 2 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
-1 & 1
\end{bmatrix}
= 
\begin{bmatrix}
0 & 3 \\
5 & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
3 & 1
\end{bmatrix}
= 
\begin{bmatrix}
5 & 5 \\
2 & -1
\end{bmatrix}
\]
Example

Calculate $AB$ where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 0 & 6 \\ 1 & 3 & -5 & 1 \\ 4 & 1 & -2 & 2 \end{bmatrix}$$

The $ij$ element of $AB$ is equal to the inner product of the $i^{th}$ row of $A$ and the $j^{th}$ column of $B$

$$AB = \begin{bmatrix} 3 & 6 & -13 & 13 \\ 26 & -5 & 0 & 32 \end{bmatrix}$$

$BA$ is not defined
Remarks (Cont’d)

2. \((A+B)^T = A^T + B^T\)

3. \((AB)^T = B^T A^T\) \hspace{1cm} \text{order reversed!}

* 4. \(AB = AC \iff B = C\) (even if \(A \neq 0\))
   
   It only implies \(A(B-C) = \phi\)

* 5. \(AB = 0 \not\Rightarrow A = 0\) or \(B = 0\)

\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 2 \\
-1 & 2 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[\phi\]

\[
\begin{array}{c}
((AB)^T)_{i,j} = [AB]_{j,i} = \sum_k a_{j,k} b_{k,i} \\
= \sum_k a_{k,i} b_{i,k} = \sum_k b_{i,k} a_{k,j} = [B^T A^T]_{i,j}
\end{array}
\]

Example

\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
3 & 2 \\
1 & 2 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 3 \\
2 & 1 \\
1 & 2
\end{bmatrix}
\]

\[
A \quad B \quad A \quad C
\]
Proof

Prove that \((AB)^T = B^T A^T\)

\[
A = \begin{bmatrix}
    a_{i1} & a_{i2} & \cdots & a_{im}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
    b_{1j} \\
    b_{2j} \\
    \vdots \\
    b_{mj}
\end{bmatrix}
\]

Multiply \(i^{th}\) row of \(A\) by \(j^{th}\) column of \(B\)

\[
\begin{bmatrix}
    A \ B
\end{bmatrix}_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}
\]

Multiply \(j^{th}\) row of \(B^T\) by \(i^{th}\) column of \(A^T\)

\[
\begin{bmatrix}
    B^T \ A^T
\end{bmatrix}_{ji} = \sum_{k=1}^{m} b_{kj} a_{ik} = \begin{bmatrix}
    A \ B
\end{bmatrix}_{ij}
\]

\[
= \begin{bmatrix}
    (B^T A^T)^T
\end{bmatrix}_{ij}
\]

\[
\Rightarrow AB = (B^T A^T)^T \quad \Rightarrow \quad (AB)^T = B^T A^T
\]
Some Special Matrices

1. Identity Matrix \( I_n \)
   - MATLAB: `eye(n,n)`

2. Diagonal Matrix \( a_{ij} = 0 \) \( \forall \ i \neq j \)
   - MATLAB: `A=diag([1 2 -1])`

3. Square Matrix
   - number of row = number of columns

4. All-Zeroes or All-Ones Matrix
   - MATLAB: `zeros(m,n)` ; `one(m,n)`

5. Symmetric Matrix \( a_{ij} = a_{ji} \)

6. (Upper and Lower) Triangular Matrix
   \( a_{ij} = 0 \) \( \forall \ j < i \)
   - For upper triangular matrix

Give Examples !!
Identity Matrix

A special matrix which plays a similar role as the number 1 in scalar multiplication

For any $n \times n$ matrix $A$, we have $A I_n = I_n A = A$

For any $n \times m$ matrix $A$, we have $I_n A = A$, and $A I_m = A$

• In Matlab: `eye(r, r)` produces an $r \times r$ identity matrix
Properties of Matrix Operations

1. \((A + B) + C = A + (B + C)\) : Associative
2. \(A + 0 = 0 + A = A\) : Identity (addition)
3. \(A + (-A) = (-A) + A = 0\) : Inverse (addition)
4. \(A + B = B + A\) : Commutative
5. \((AB)C = A(BC)\) : Associative
6. \(A(B + C) = AB + CA\) : Distributive
7. \((B + C)A = BA + CA\) : Distributive
8. \((A^T)^T = A\)
9. \(A.I = I.A = A\) : Identity (multiplication)

Remark: matrix multiplication was defined in such a way that properties 5, 6, 9 (and others) hold!
Properties of Matrices

Trace of a matrix is defined only for a square matrix

Definition: \[ tr(A) = \sum_{i=1}^{m} a_{ii} \]

MATLAB: `trace(A)`

Properties: prove 1-3 using definition of trace(.)

1. \[ tr(A + B) = tr(A) + tr(B) \]
2. \[ tr(kA) = k \cdot tr(A) \quad k \text{ is a scalar} \]
3. \[ tr(A^T) = tr(A) \]
4. \[ tr(BA) = tr(AB) \quad \text{however } AB \neq BA \]

Proof: 
\[ (AB)_{ii} = \sum_{k=1}^{m} a_{ik} b_{ki} \implies tr(AB) = \sum_{i=1}^{m} \sum_{k=1}^{m} a_{ik} b_{ki} \]

\[ (BA)_{ii} = \sum_{k=1}^{m} b_{ik} a_{ki} \implies tr(BA) = \sum_{i=1}^{m} \sum_{k=1}^{m} a_{ki} b_{ik} \]

Hint:
Replace k by i
Matrix Representation for System of Linear Equations

Going back to the system of Linear Equations:

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{bmatrix}
\]

or \( Ax = b \)

A = coefficient matrix

x = Unknown vector

How to solve for x?

\( x = A^{-1}b \)  

( Matrix generalization of \( x = b/a \) )

How to define a matrix inverse? coming soon!
How to Solve a System of Linear Equations?

Remark: It is challenging to solve this system of equations since the equations are coupled, however, it becomes easy to solve

1. if $a_{ij} = 0 \quad \forall i \neq j$ \quad $\Rightarrow$ A is diagonal

$$\Rightarrow x_i = \frac{b_i}{a_{ii}} \quad \text{trivial: } a_{ii} \neq 0$$

2. if $a_{ij} = 0 \quad \forall i > j$ \quad $\Rightarrow$ A is upper triangular

Example:

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\Rightarrow 2x_2 = -2 \quad \Rightarrow x_2 = -1$$

$$\Rightarrow x_1 + 2x_2 = -1 \quad \Rightarrow x_1 - 2 = -1$$

$$\Rightarrow x_1 = 1 \quad \text{(back substitution)}$$
System of Linear Equations

$$ Ax = b $$

- **homogenous** $b = 0$
- **non-homogenous** $b \neq 0$

**Inconsistent**
No Solution

**Consistent**

- **Unique Solution**
- **Infinite Number of Solutions**

**Remark:** We can never have 2 solutions (either 0, 1 or infinite !)

**Proof:** Suppose $Ax_1 = b$ & $Ax_2 = b$

Consider $x_3 = \alpha x_1 + (1 - \alpha)x_2$

$$ Ax_3 = \alpha Ax_1 + (1 - \alpha)Ax_2 = \alpha b + (1 - \alpha)b = b $$

$$ \Rightarrow x_3 \text{ is also a solution } \forall \alpha \in R \text{ (infinite solutions) } $$
Examples in Three Dimensions

Unique solution

Three planes $A$, $B$, and $C$ intersect at a single point $P$. $P$ corresponds to a unique solution.

No solution

Planes $A$, $B$, and $C$ have no points in common. There is no solution.

Many solutions

Three planes $A$, $B$, and $C$ intersect in a line $PQ$. Any point on a line is a solution.
Examples in Two Dimensions

Unique solution

\[ x + y = 5 \]
\[ 2x - y = 4 \]

Write as \( y = -x + 5 \) and \( y = 2x - 4 \). The lines have slopes \(-1\) and \(2\), and \(y\)-intercepts \(5\) and \(-4\). They intersect at a point, the solution. There is a unique solution, \(x = 3, y = 2\).
No solution

\[-2x + y = 3\]
\[-4x + 2y = 2\]

Write as \(y = 2x + 3\) and \(y = 2x + 1\). The lines have slope 2, and \(y\)-intercepts 3 and 1. They are parallel. There is no point of intersection. No solution.
Many solutions

\[ 4x - 2y = 6 \]
\[ 6x - 3y = 9 \]

Each equation can be written as \( y = 2x - 3 \). The graph of each equation is a line with slope 2 and \( y \)-intercept \(-3\). Any point on the line is a solution.

Many solutions