Phase Noise in Asynchronous SC-FDMA Systems: Performance Analysis and Compensation

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Abstract—Carrier frequency offset (CFO) and phase noise (PN) are major oscillator impairments in direct-conversion transceivers, and single-carrier frequency-division-multiple access (SC-FDMA) is the uplink transmission scheme in the long-term evolution (LTE) standard. We derive a new analytical expression for the normalized mean square error (NMSE) in asynchronous SC-FDMA systems under CFO and joint transmit-receive PN. The derived NMSE expression reveals an interesting cross-layer relationship between the subcarrier mapping scheme at the medium-access control (MAC) layer and the immunity to CFO and PN at the physical layer. Furthermore, we propose an iterative reduced-complexity joint decoding and PN compensation scheme which does not require any pilots in PN tracking and exploits the low-pass nature of the PN process without assuming a specific PN model. Simulation results show the effectiveness of our proposed digital compensation scheme in PN mitigation.

Index Terms—Phase noise, SC-FDMA.

I. INTRODUCTION

Phase noise (PN) is one of the major radio-frequency (RF) front-end impairments [1], [2] resulting from the imperfections and inaccuracies of the crystal oscillator’s fabrication process. Several techniques [3]–[7] have been proposed to estimate and compensate for PN in orthogonal-frequency-division multiplexing (OFDM) systems. The impact of CFO and PN on channel estimation is investigated in [8]. In [5]–[7], iterative detection and PN compensation techniques are proposed with pilot subcarriers used to estimate the PN-induced common phase-error (CPE) as an initialization for the PN compensation process. In [6], each OFDM symbol is divided into sub-blocks where the PN process is assumed quasi-static over each sub-block. A Kalman filter for PN tracking is proposed in [7] where knowledge of the PN statistical model is assumed.

A common assumption in all of the above-cited approaches is the availability of pilot subcarriers multiplexed in each data OFDM symbol. Unfortunately, this assumption is not valid for SC-FDMA systems adopted in LTE uplink [9] to maintain low peak-to-average power ratio (PAPR). Instead, at the beginning of each uplink frame, the pilots are transmitted in a dedicated training symbol followed by six data SC-FDMA symbols void of pilots in the LTE-uplink [9]. These training symbols are shown to be sufficient for channel tracking even under high-mobility scenarios. However, they may not be sufficient for tracking other time-varying impairments such as PN whose samples vary from one time instant to another within a single SC-FDMA symbol. Hence, PN tracking is a challenging task for the LTE uplink where consecutive training symbols are six symbols apart with no pilots inserted in the data symbols in between. In OFDM systems, the PN tracking task is easier thanks to the pilot subcarriers multiplexed with data subcarriers. In [10], maximum-likelihood estimation of CFO in the LTE uplink is derived.

The PN maximum-a-posteriori (MAP) estimate is derived in [11] for single-carrier single-user systems employing turbo equalization without pilots. The MAP solution was shown in [11] to be a second-order phase-locked loop for constant-amplitude modulation (such as BPSK and QPSK). Furthermore, the PN statistical model was assumed available and the extension to higher-order modulation is not straightforward. In this paper, we tackle the problem of non-pilot-aided PN compensation in SC-FDMA systems without knowledge of the PN model. We propose an iterative joint data decoding and PN compensation scheme which exploits the low-pass nature of the PN process without assuming a priori knowledge about the exact PN model to avoid model mismatch problems. In addition, no restrictions on the constellation size or shape are imposed. Unlike existing approaches, by writing the system model in an alternative form, we were able to move the channel equalizer out of the iterative loop to reduce the complexity. We also show that our proposed PN compensation technique is robust to channel estimation errors. Comparison with [11] shows that the performance of our approach is close to that of the MAP solution with significantly reduced complexity and wider applicability to different constellation sizes. Currently, accurate oscillators are used in user equipments (UEs) and, hence, the PN impact can be safely neglected. However, the main motivation behind our paper is to relax the design requirements of the UE oscillator allowing for inexpensive oscillators where PN is compensated at the baseband level using our proposed PN compensation method.

We derive a new analytical expression for the NMSE under joint transmit-receive PN and carrier frequency offset (CFO). Furthermore, based on the derived NMSE expression, we compare the robustness of both localized and distributed subcarrier mapping schemes under CFO and PN. The impact
of PN was investigated in [12] for single-carrier single-user systems. However, the derived expression in [12] assumed no channel equalization and had functional dependence on the instantaneous channel realization which varies randomly over time. However, in this paper, we include channel equalization effects and take the statistical expectation over the channel realization. The exact expectation of the ratio rather than the approximate ratio of expectations is derived. Furthermore, the SC-FDMA performance has been recently analyzed [13] under receive-only PN. However, transmit PN is more serious and requires more attention because the UE oscillator is expected to be less accurate than that of the Base Station (BST).

The rest of this paper is organized as follows. The system model is described in Section II and the NMSE expression is given in Section III with the detailed derivation relegated to the appendix. In Section IV, we simplify the derived NMSE expression and analytically compare the performances of localized and distributed subcarrier mapping schemes under CFO and PN. Our proposed PN compensation scheme is described in Section V. Simulation results are presented in Section VI and the paper is concluded in Section VII.

Notations: Lower and upper-case bold letters denote vectors and matrices, respectively, and \( \mathbf{A}(m,n) \) denotes the element in the \( m \)-th row and \( n \)-th column of \( \mathbf{A} \). The \( j \)-th element of \( \mathbf{a} \) is denoted by \( a(j) \), and \( \mathbf{a}(j:k) \) denotes the portion of a starting at index \( j \) and ending at index \( k \geq j \). \( \mathbf{I} \) and \( \mathbf{F} \) denote, respectively, the identity and Fast Fourier Transform (FFT) matrices, and their subscripts denote theirs sizes, and \( \mathbf{0} \) is the all-zero matrix. For matrices, \( \mathbf{A} \equiv \mathbf{FA}
 F^H \). Also, \( (\cdot)^H \), \( (\cdot)^* \), and \( (\cdot)^T \) denote the matrix Hermitian, complex conjugate, and transpose operations, respectively, while \( (\cdot)^{-1} \) denotes both scalar reciprocal and matrix inverse. \( E[\cdot], |\cdot| \), and \( [\cdot] \) denote, respectively, statistical expectation, absolute, ceiling and floor functions. Also, \( \text{diag}(\ldots) \) denotes a diagonal matrix whose diagonal is the argument; while \( \text{diag}(\mathbf{A}) \) denotes a column vector containing the diagonal entries of \( \mathbf{A} \).

II. SYSTEM MODEL

A. Signal Model

We consider the uplink SC-FDMA transmission scenario with one receive antenna at BST and \( N_u \) single-antenna users. The total number of subcarriers is \( N \) where each user is exclusively assigned \( M < N \) subcarriers such that each subcarrier is not assigned to more than one user in the same time slot. The subcarrier mapping of the \( l \)-th user is defined by the \( N \times M \) matrix \( \mathbf{S}_l \) whose elements are zeros except for a single ‘1’ in each column. The indices of the rows containing these ones are denoted by the set \( J_l \) and correspond to the locations of the assigned subcarriers, and the cardinality of \( J_l \) is \( M \), \( \forall l \). The \( l \)-th SC-FDMA time-domain (TD) symbol generated by the \( l \)-th user is given by

\[
\mathbf{g}_l^t = \mathbf{P}^H_l \mathbf{S}_l \mathbf{F}_M \mathbf{x}_l^t
\]

where \( \mathbf{x}_l^t \) is the \( M \times 1 \) data vector generated by the \( l \)-th user in the \( l \)-th SC-FDMA symbol with \( E[\mathbf{x}_l^t \mathbf{x}_l^T] = \eta_l \mathbf{I}_M \) and \( \eta_l \) is the received power from the \( l \)-th user. Then, a cyclic-prefix of length \( L_p \) greater than or equal to the channel memory is appended to the vector \( \mathbf{g}_l \) before transmission.

B. PN and CFO Models

CFO results from inaccuracies of the crystal oscillators used at the transmitter and receiver causing the carrier frequency to deviate from its nominal value. Furthermore, the power spectrum density (PSD) of the generated carrier signal typically follows the Lorentzian shape [14] causing energy leakage from each subcarrier into its neighbors resulting in inter-carrier interference (ICI). This impairment causes the oscillator phase to be noisy where this noise follows the Wiener random process. In the presence of CFO and joint transmit-receive PN, the received TD signal is given by

\[
y^t = \mathbf{P}_r^H \sum_{l=1}^{N_u} \mathbf{Q}_l^t \mathbf{H}_r^l \mathbf{P}_l^t \mathbf{g}_l^t + z^t
\]

where \( \mathbf{H}_r^l \) is the \( N \times N \) circulant channel matrix whose first column contains the zero-padded channel impulse response (CIR) between the \( l \)-th user and the BST, and \( z^t \) is the complex additive white Gaussian noise (AWGN) vector with a single-sided PSD of \( N_0 \) Watt/Hz. The CFO between the \( l \)-th user and the BST is modeled by the \( N \times N \) diagonal matrix

\[
\mathbf{Q}_l^t \equiv \text{diag} \left( \left\{ \exp \left( \frac{j2\pi m \alpha_l}{N} \right) \right\} \right)_{m = (N + L_p), \ldots, N - 1}
\]

with \( \alpha_l \equiv \Delta f_l / f_{sub} \) where \( \Delta f_l \) is the difference in Hz between the carrier frequencies of the \( l \)-th user and the BST and \( f_{sub} \) denotes the subcarrier frequency spacing in Hz. The transmit and receive PN processes are modeled [14] by the multiplication of the TD signals by the diagonal matrices \( \mathbf{P}_i^t \) and \( \mathbf{P}_r^t \), respectively, given by

\[
\mathbf{P}_i^t \equiv \text{diag} \left( \left\{ \exp \left( j \phi_{0i}^t \right), \exp \left( j \phi_{1i}^t \right), \ldots, \exp \left( j \phi_{N_i}^t \right) \right\} \right)
\]

\[
\mathbf{P}_r^t \equiv \text{diag} \left( \left\{ \exp \left( j \phi_{0r}^t \right), \exp \left( j \phi_{1r}^t \right), \ldots, \exp \left( j \phi_{N_r}^t \right) \right\} \right)
\]

where \( \phi_{0i}^t \) and \( \phi_{0r}^t \) are the PN samples perturbing the transmitted and received signals, respectively, at the \( n \)-th sample of the \( l \)-th SC-FDMA symbol. The phase noise is modeled by the following first-order auto-regressive [15] processes

\[
\phi_{n+1}^\prime = \phi_n^\prime + \epsilon_n^\prime, \quad \phi_{n+1}^r = \phi_n^r + \epsilon_n^r
\]

where \( \epsilon_n^\prime \sim \mathcal{N}(0, \sigma\) and \( \epsilon_n^r \sim \mathcal{N}(0, \sigma^2) \) are independent Gaussian-distributed random variables. Without loss of generality, we assume that \( \phi_{0i}^\prime = \phi_{0r}^\prime = 0, \forall l \). The parameters \( \beta_i \) and \( \beta_r \) denote the two-sided 3-dB linewidths of the Lorentzian-shaped PSD of the oscillators feeding the \( l \)-th user and the BST, respectively. For simplicity, we drop the SC-FDMA symbol index, \( t \), in Sections III and IV-B where we analyze the performance under CFO and PN.

III. NMSE DERIVATION

Denoting by \( k \) the index of the user of interest, we compute its decision statistic in four steps. First, we apply the \( N \)-point FFT to \( y^t \). Second, we select its assigned \( M \) subcarriers by applying the selection matrix \( \mathbf{S}_k^H \). Third, we apply frequency-domain (FD) equalization to mitigate the channel frequency-selectivity effects. Fourth, we apply the \( M \)-point inverse FFT

\footnotetext[1]{We assume this model to simplify the NMSE derivation. However, we do not assume it for PN compensation in Section V.}
to go back to TD for data detection. We do not use joint-subcarrier processing at the third step, i.e., the equalization matrix is diagonal; therefore, the second and third steps can be interchanged. Following this procedure, we write the decision statistic of the $k$-th user as follows

$$
\hat{x}_k = U_{kk} x_k + \sum_{l \neq k} U_{kl} x_l + \hat{z}
$$

with the three terms in (6) given by

$$U_{kk} = F_h^H S_k^H D_k P_k Q_k H_k P_k S_k F_M
$$
$$U_{kl} = F_h^H S_k^H D_k P_r Q_k H_p P_l S_l F_M
$$
$$\hat{z} = F_h^H S_k^H D_k F_N z
$$

where $D_k \equiv \text{diag} \left( \left\{ \text{Tr}_r(\mathbf{h}^H_{(n,n)} + \gamma^{-1}_k) \right\}_{n=0}^{N-1} \right)$ is the diagonal minimum-mean squared-error (MMSE) FD equalization matrix of the $k$-th user with $\gamma_k \triangleq \frac{\sigma^2}{\sigma^2}$ being the signal-to-noise ratio (SNR) of the $k$-th user. As defined in Section I, $P_r \equiv F_N P_r F_N^H$, $Q_k \equiv F_N Q_k F_N^H$, and $H_k \equiv F_N H_k F_N^H$. Also, $H_k \equiv F_N H_k F_N^H$ is the $N \times N$ diagonal FD channel matrix. Note that the diagonal and off-diagonal entries of the matrix $U_{kk}$ correspond to the desired and ICI terms, respectively. The matrix $U_{kl}$ corresponds to the inter-user interference (IUI) from the $l$-th user into the $k$-th user. The MMSE of the $k$-th user is defined as follows

$$\text{NMSE}_k \triangleq \sqrt{\frac{\mathbb{E}[\text{Tr}(e^H)]}{\eta_k}} = \sqrt{\frac{\mathbb{E}[\text{Tr}(e^H)]}{\eta_k}} \times 100 \quad (\%)$$

where the error vector $e \triangleq \hat{x}_k - x_k$ and $\text{Tr}()$ denotes the matrix trace. Assuming that the data vectors of different users are independent with zero mean, that the user’s data symbols are independent, and that the data and noise vectors are independent, we write

$$\mathbb{E}[\text{Tr}(e^H)] = \eta_k \mathbb{E}[\text{Tr}(U^H_{kk} (U^r_{kk})^H)]
$$

$$+ \sum_{l \neq k} \eta_l \mathbb{E}[\text{Tr}(U_{kl} U^H_{kl})] + \mathbb{E}[\text{Tr}(\hat{z}^H\hat{z})]
$$

where $U^H_{kk} \triangleq U_{kk} - I_M$. The first term, denoted by $t_1$, in (8) is evaluated as follows

$$t_1 = \mathbb{E}[\text{Tr}(U^H_{kk} (U^r_{kk})^H)] = M - 2\text{Re} \left( \mathbb{E}[\text{Tr}(U_{kk})] \right)
$$

$$+ \mathbb{E}[\text{Tr}(U_{kk} U^H_{kk})]
$$

$$\triangleq t_{1,1}
$$

(9)

where $\text{Re}(.)$ denotes the real part of its argument. In the sequel, we derive the quantities $t_{1,1}$, $t_{1,2}$, $t_{2,2}$, and $t_3$ in (8) and (9).

Using the identity $\text{Tr}(AB) = \text{Tr}(BA)$ for any two matrices $A$ and $B$, we start with $t_{1,1}$ in (9) and write

$$t_{1,1} = \mathbb{E}[\text{Tr}(U_{kk})] = \mathbb{E}[\text{Tr}(P_k Q_k H_k P_k S_k D_k)]
$$

(10)

where $S_k \triangleq S_k S_k^H$ is a diagonal matrix. Using the fact that diagonal matrices are commutative under multiplication and assuming that the channel and PN processes are statistically independent, we write

$$\mathbb{E}[\text{Tr}(U_{kk})] = \mathbb{E} \left[ \text{Tr} \left( \mathbb{E} \left[ P_r \right] Q_k H_k \mathbb{E} \left[ P_k \right] D_k S_k \right) \right]
$$

(11)

Since both $H_k$ and $D_k$ are diagonal matrices, we rewrite (11) as follows

$$\mathbb{E}[\text{Tr}(U_{kk})] = \text{Tr} \left( \mathbb{E} \left[ P_r \right] Q_k
$$

$$\left( \mathbb{E} \left[ \text{diag}(H_k) \text{diag}(D_k) \right]^T \otimes \mathbb{E} \left[ P_k \right] \right) S_k \right)
$$

(12)

where $\mathbb{E}[]$ and $\text{Tr}()$ are linear operators and, hence, can be exchanged. Furthermore, the notation $\otimes$ denotes the element-wise Hadamard product. We assume that the CIR follows the well-known zero-mean circular symmetric complex Gaussian distribution and so does $H_k(n,n)$ which is a weighted summation of the CIR taps.

**Lemma 1.** Let $f(x)$ and $g(y)$ denote two functions of the jointly Gaussian random variables $x$ and $y$, respectively, with the operator $\mid$ denoting the conditional expectation. Then,

$$E_{x,y}[f(x)g(y)] = E_x E_{y|x}[f(x)g(y)] = E_x \left[ f(x) E_{y|x}[g(y)] \right]
$$

$$= \int f(x) \int g(y)p(y|x)dy\,dx
$$

(13)

where $p(y|x)$ is the conditional probability density function of $y$ given $x$. Since $x$ and $y$ are jointly Gaussian, $p(y|x)$ is also Gaussian whose conditional mean and variance [16] are

$$E[y|x] = E[y] + \frac{\text{cov}[y,x]}{\text{var}[x]} (x - E[x]),
$$

$$\text{var}[y|x] = \text{var}[y] - \frac{[\text{cov}[y,x]]^2}{\text{var}[x]}
$$

(14)

where $\text{var}[x] \triangleq \mathbb{E}(|x - E[x]|^2)$ and $\text{cov}[y,x] \triangleq \mathbb{E}((y - E[y])(x - E[x]))^*$.

Denoting $H_k(m,m)$ and $H_k(n,n)$ by $y$ and $x$, respectively, we exactly (unlike [13] and [17]) compute the entries of the matrix $T_k \triangleq \mathbb{E} \left[ \text{diag}(H_k) \text{diag}(D_k) \right]^T$ as follows

$$T_k(m,n) = E_{x,y} \left[ \frac{xy^*}{|x|^2 + \gamma^{-1}_k} \right] = E_x \left[ \frac{xy^*}{|x|^2 + \gamma^{-1}_k} \right]
$$

$$= E_x \left[ x^* E_{y|x}[y] \right], \quad \forall \, m, n
$$

(15)

Since $H_k(m,m)$ and $H_k(n,n)$ are jointly Gaussian, we use Lemma 1 and write

$$E_{y|x}[y] = E[y] + \frac{E[xy^*]}{E|x|^2} (x - E[x]) = \frac{R_k(m,n)}{R_k(n,n)} x
$$

(16)

$$T_k(m,n) = \frac{R_k(m,n)}{R_k(n,n)} \int \frac{|z|^2}{|z|^2 + \gamma^{-1}_k} \exp(-z/E[z])dz
$$

$$= \frac{R_k(m,n)}{R_k(n,n)} \left[ 1 + p_k^\mu \exp(p_k^\nu) \text{Ei}(-p_k^\nu) \right], \quad \forall \, m, n
$$

(17)
where $z \triangleq |x|^2$ is exponentially distributed and $\mathbb{E}[z] = \mathbb{E}\left[|H_k(n,n)|^2\right] = R_k(n,n)$. Furthermore, $p_{k}^{m} \triangleq (\gamma_k R_k(n,n))^{-1}$, $R_k(m,m) \triangleq \mathbb{E}\left[H_k(m,m)H_k^*(n,n)\right]$ is the FD channel auto-correlation matrix, and $\text{Ei}(-p_{k}^{m}) \triangleq \int_{-p_{k}^{m}}^{\infty} \frac{\exp(x)}{x} dx$ is the exponential integral function [18]. Then, the final expression of $t_{1A}$ is given by

$$t_{1A} = \text{Tr} \left( E[P_{r/k}] Q_k (T_k \circ E[P_k]) S_k \right)$$

$$E[P_{r/k}] = F_N \text{diag} \left( \exp \left( -\frac{m \sigma_r^2/k}{2} \right) \right)_{m=0}^{N-1} F_N^H$$

where we used the fact that $\mathbb{E}[e^{jy}] = e^{-\sigma_y^2/2}$ [19] for $y \sim N(0, \sigma^2)$. Next, we evaluate the quantity $t_{1B}$ in (9) as follows

$$t_{1B} \triangleq \mathbb{E} \left[ \text{Tr} \left( U_{kk} U_{kk}^H \right) \right] = \mathbb{E} \left[ \text{Tr} \left( P_r Q_k H_k H_k^H Q_k^* P_r^* D_k D_k^H S_k \right) \right]$$

$$W_k \triangleq \mathbb{E} \left[ P_k S_k F_k^H \right] = F_N \mathbb{E} \left[ P_k \left( F_N^H S_k F_N \right) P_k^H \right] F_N^H$$

$$= \mathbb{E} \left[ N \sigma_s^2 F_N \circ V_k \right] F_N^H$$

and $V_k \triangleq \mathbb{E} \left[ \text{diag}(P_k) \text{diag}(P_k)^H \right]$ whose entries are given in (35). Inspecting the structure of $W_k$ in (21), we find that $F_N^H S_k F_N$ is a Toeplitz matrix [20] because $V_k$ is Toeplitz and $F_N^H S_k F_N$ is circulant. Furthermore, Toeplitz matrices with large sizes can be accurately approximated as circulant matrices [20]; hence, for large FFT sizes, the matrix $W_k$ can be approximated by its main diagonal while setting the off-diagonal entries to zero. Using this approximation, we interchange the approximately diagonal matrix $W_k$ and the diagonal matrix $H_k$ and proceed as follows

$$t_{1B} \approx \mathbb{E} \left[ \text{Tr} \left( P_r Q_k H_k H_k^H Q_k^* P_r^* D_k D_k^H S_k \right) \right]$$

$$= \mathbb{E} \left[ \text{Tr} \left( P_r Q_k W_k \right) \left( \left( Q_k^* P_r^* \right) \circ Y_k \right) S_k \right]$$

$$Y_k \triangleq \mathbb{E} \left[ \text{diag}(H_k H_k^H) \right] \text{diag}(D_k D_k^H)]^{T}$$

where $Y_k(m,n) = \mathbb{E} \left[ \left| H_k(m,m) \right|^2 \left| H_k(n,n) \right|^2 \right]$ and $Y_k$ is diagonal (Unlike [17]) computed in (31) using the relation $\mathbb{E}[y^2|x] = \text{var}[y|x] + [E[y|x]]^2$ added to Lemma 1 which yields

$$\mathbb{E} \left[ \left| H_k(m,m) \right|^2 \right] = \frac{R_k(m,m)^2}{|R_k(m,m)|^2} \left| H_k(n,n) \right|^2$$

$$+ \frac{R_k(m,m)R_k(n,n) - |R_k(m,m)|^2}{|R_k(n,n)|^2}$$

Since $W_k$ is approximated by a diagonal matrix, it can be moved and multiplied by $Y_k$ as follows

$$t_{1B} \approx \mathbb{E} \left[ \text{Tr} \left( P_r Q_k \left( Q_k^* P_r^* \right) \circ W_k Y_k \right) S_k \right]$$

$$= \mathbb{E} \left[ \text{Tr} \left( (C_k \circ C_k^*) L_k S_k \right) \right]$$

$$= \sum_{n \in J_k} \sum_{m=0}^{N-1} \mathbb{E} \left[ \left| C_k \left( 1, (m-n) \right) \right|^2 \right] L_k(m,n)$$

where $C_k \triangleq P_r Q_k$, $L_k \triangleq W_k Y_k$, and $(m-n) \mod N = (m - n)$ mod $N$ is the mode-$N$ operation which originates from the circulant structure of the matrix $C_k$. Furthermore,

$$\mathbb{E} \left[ C_k \left( (m-n) \right) \right] = \frac{1}{\sqrt{N}} f_{C_k}^{(m-n)} \circ \text{diag} \left( P_r Q_k \right)$$

$$\mathbb{E} \left[ \left| C_k \left( (m-n) \right) \right|^2 \right] = \frac{1}{N} f_{C_k}^{(m-n)}$$

$$\times \mathbb{E} \left[ \text{diag} \left( P_r Q_k \right) \text{diag} \left( P_r Q_k \right)^H \right] f_{(m-n)}^H$$

$$= \mathbb{E} \left[ \text{diag} \left( P_r Q_k \right) \text{diag} \left( P_r Q_k \right)^H \right]$$

Then, it can be easily shown that the entries of $V_r$ are those given in (28). Substituting (26) back into (24), we get the final expression of $t_{1B}$ as follows

$$t_{1B} \approx \frac{1}{N} \sum_{n \in J_k} \sum_{m=0}^{N-1} \mathbb{E} \left[ f_{C_k}^{(m-n)} \circ V_r \right] f_{C_k}^{(m-n)} \circ L_k(m,n)$$

$$V_r(m,n) = \exp \left( -\frac{|m-n| \sigma_s^2}{2} + \frac{j2\pi(m-n)\alpha_k}{N} \right)$$

$$L_k = F_N \left( F_N^H S_k F_N \circ V_r \right) Y_k$$

$$V_t(m,n) = \exp \left( -\frac{|m-n| \sigma_s^2}{2} \right)$$

$$Y_t(m,n) = \frac{|R_k(m,n)|^2}{|R_k(n,n)|^2} \times (1 + p_k^r + p_k^s (p_k^s + 2) \text{exp}(p_k^s) \text{Ei}(-p_k^s))$$

$$\forall m, n = 0, 1, \ldots, N - 1. \text{ Next, we derive the quantity } t_2 \text{ in (8) for the channels experienced by different users are independent.}$$

$$t_2 \triangleq \mathbb{E} \left[ \text{Tr} \left( U_{kk} U_{kk}^H \right) \right]$$

$$= \mathbb{E} \left[ \text{Tr} \left( P_r Q_k H_k H_k^H Q_k^* P_r^* D_k D_k^H S_k \right) \right]$$

$$\mathbb{E} \left[ \text{diag}(H_t H_t^H) \left( \text{diag}(D_t D_t^H) \right)^T \right]$$

$$D_h(n,n) = \frac{|H_k(n,n)|^2}{|H_k(n,n)|^2 + \gamma_k^{-1}}$$

$$= \frac{-1}{|R_k(n,n)|^2} \left( (1 + 1 + p_k^h) \exp(p_k^h) \text{Ei}(-p_k^h) \right)$$

Exploiting that $H_t$ is diagonal, we re-write (32) as follows

$$t_2 = \text{Tr} \left( E \left[ P_r \left( W_t \circ E \left[ \text{diag}(H_t) \text{diag}(H_t)^H \right] \right) \right] \right)$$

$$= \text{Tr} \left( F_N P_r Q_s F_N^H G_t F_N Q_t^H P_t^H D_t S_k \right)$$

$$= \text{Tr} \left( F_N \left( F_N^H G_t F_N \circ E \left[ \text{diag}(P_r Q_t) \text{diag}(P_r Q_t)^H \right] \right) F_N^H D_t S_k \right)$$

Observing that $R_t = \mathbb{E} \left[ \text{diag}(H_t) \left( \text{diag}(H_t)^H \right)^T \right]$ and defining $G_t \triangleq W_t \circ R_t$, we write

$$t_2 = \text{Tr} \left( E \left[ P_r Q_s F_N^H G_t F_N Q_t^H P_t^H D_t S_k \right] \right)$$

$$= \text{Tr} \left( E \left[ F_N P_r Q_s F_N^H G_t F_N Q_t^H P_t^H D_t S_k \right] \right)$$

$$= \text{Tr} \left( F_N \left( F_N^H G_t F_N \circ E \left[ \text{diag}(P_r Q_t) \text{diag}(P_r Q_t)^H \right] \right) F_N^H D_t S_k \right)$$
where the last equality holds because the matrix $P_iQ_i$ is diagonal. Inspecting the definition of $V_r$ in (26), we write

$$t_2 = \text{Tr} \left( F_N \left( F_N^H G_i F_N \circ V_r \right) F_N^H D_k S_k \right) \triangleq \text{Tr} \left( W_r D_k S_k \right)$$

$$= \sum_{n \in J_k} W_r(n,n) D_k(n,n)$$

Substituting the entries of $D_k$ from (33) into (36), we get the final expression of $t_2$ as follows

$$t_2 = \sum_{n \in J_k} -\frac{W_r(n,n)}{R_k(n,n)} \left( 1 + (1 + p_k^n) \exp(p_k^n) \text{Ei}(-p_k^n) \right)$$

where the diagonal matrices $W_r$ and $W_l$ are given by

$$W_r = F_N \left( (F_N^H (W_l \circ R_l) F_N) \circ V_r \right) F_N^H$$

$$W_l = F_N \left( (F_N^H S_l F_N) \circ V_l \right) F_N^H$$

Finally, we evaluate $t_3$ (which is the noise contribution to the NMSE in (8)) as follows

$$t_3 \triangleq \mathbb{E} \left[ \text{Tr} (zz^H) \right] = N_o \text{Tr} (D_k S_k) = N_o \sum_{n \in J_k} D_k(n,n)$$

Substituting the entries of $D_k$ from (33) into (38), we get the final expression of $t_3$ as follows

$$t_3 = \sum_{n \in J_k} -\frac{N_o}{R_k(n,n)} \left( 1 + (1 + p_k^n) \exp(p_k^n) \text{Ei}(-p_k^n) \right)$$

It is worth mentioning that, unlike [17], the entries of $T_k$ and $Y_k$ are computed exactly using Lemma 1.

IV. NMSE Expression Simplification and Subcarrier Mapping Impact

A. NMSE Expression Simplification

To gain insight into the NMSE expression given in Section III, we simplify it by considering the high-SNR scenario with only transmit PN$^2$, i.e., $P_r = Q_k = I_N$, $\forall k$, and $D_k H_k \approx I_N$. In uplink transmissions, receive PN is significantly smaller than transmit PN since the BST oscillator quality is typically better than that of the user terminal. Under the above assumptions, we re-write the NMSE expression terms as follows

$$t_{1A} = \mathbb{E} \left[ \text{Tr} \left( H_i H_i^H D_k S_k \right) \right] = \mathbb{E} \left[ \text{Tr} \left( P_k S_k D_k H_k \right) \right]$$

$$= N \sum_{m=0}^{N-1} \exp \left( -m \sigma_k^2 \right) \approx M$$

where the second equality follows from $\text{Tr} (AB) = \text{Tr} (BA)$ and the approximation $\frac{1}{N} \sum_{m=0}^{N-1} \exp \left( -m \sigma_k^2 \right) \approx 1$ which holds under practical values of $\sigma_k^2$ ranging from $10^{-6}$ to $10^{-3}$. Next, we re-write $t_{1B}$ as follows

$$t_{1B} = \mathbb{E} \left[ \text{Tr} \left( H_i W_k H_i^H D_k S_k \right) \right]$$

$$= \mathbb{E} \left[ \text{Tr} \left( W_k H_i H_i^H D_k S_k \right) \right]$$

$$= \text{Tr} \left( W_k S_k \right) \sum_{n \in J_k} W_k(n,n)$$

where $W_k$ is defined in (21) and the commutative property of diagonal matrices is used. In Section III, we showed that $W_k$ can be approximated by a diagonal matrix. Inspecting (21) and (30), we observe that the diagonal of $W_k$ is the circular convolution between the diagonal of $S_k$ and the FFT of the sequence $\{ \exp \left( \frac{-\text{Ei}(p_k^n)}{2} \right) \}_{m=-\lfloor N/2 \rfloor}^{\lceil N/2 \rceil}$ which is nothing but the Lorentzian spectrum. Furthermore, the diagonal of $S_k$ is nonzero only at the subcarrier indices of the $k$-th user. We use the FFT property that element-wise multiplication in TD corresponds to circular convolution in FD. Recalling from Section II-B that the PN PSD is Lorentzian-shaped, we conclude that the sequence $\{ W_k(n,n) \}$ in (41) is the circular convolution between the user’s spectrum and the PN spectrum. This circular convolution quantifies the PN-induced intra-user interference. Similarly, we re-write $t_2$ as follows

$$t_2 = \mathbb{E} \left[ \text{Tr} \left( \bar{H_i} W_l \bar{H_i}^H D_k S_k \right) \right] = \mathbb{E} \left[ \text{Tr} \left( W_l \bar{H_i} H_i^H D_k S_k \right) \right]$$

$$\simeq \sum_{n \in J_k} W_l(n,n) D_k(n,n) \mathbb{E} \left[ \bar{H_i}^H(\bar{H_i}(n,n))^2 \right]$$

$$= \sum_{n \in J_k} W_l(n,n) - \frac{R_l(n,n)}{R_k(n,n)} (1 + (1 + p_k^n) \exp(p_k^n) \text{Ei}(-p_k^n))$$

$$\simeq \sum_{n \in J_k} W_l(n,n) - \frac{R_l(n,n)}{R_k(n,n)} \text{Ei}(-p_k^n)$$

where $D_k(n,n)$ is given in (33) and the last approximation follows from the high-SNR assumption where $p_k^n \rightarrow 0$ and $\text{Ei}(-p_k^n) \rightarrow -\infty$. If the channels experienced by different users are identically distributed, i.e., $R_l = R_k$, then

$$t_2 \simeq \sum_{n \in J_k} -W_l(n,n) \text{Ei}(-p_k^n)$$

Similar to $W_k$, the sequence $\{ W_l(n,n) \}$ represents the circular convolution between the PN spectrum and the $l$-th user spectrum. Since the PN spectrum is not impulse, spectrum energy leakage of the $l$-th user causes inter-user interference. The quantity $t_2$ quantifies the $l$-th user’s energy leakage at the $k$-th user’s subcarriers. Applying the high-SNR approximation to $t_3$, we write the simplified NMSE expression as follows

$$\text{NMSE}_k^2 \simeq \frac{1}{M} \left( \sum_{n \in J_k} W_k(n,n) \right)^2 - 1$$

$$+ \frac{1}{M} \sum_{l \neq k} \sum_{n \in J_k} u_l W_l(n,n) + \frac{1}{M} \sum_{n \in J_k} -p_k^n \text{Ei}(-p_k^n)$$

where $u_l = -\frac{R_l(n,n)}{R_k(n,n)} \text{Ei}(-p_k^n)$. Furthermore, $\text{Ei}(-p_k^n)$ can be approximated by $\text{Ei}(-p_k^n) \approx 0.577 + \log(p_k^n) - p_k^n$ for $p_k^n \leq 0.3$ or, equivalently, for $\gamma_k \geq 5$ dB where $\log(.)$ is the natural logarithm function.

B. Localized versus Distributed Subcarrier Mapping

In light of the simplified NMSE expression in (44), we investigate the effect of the user’s subcarrier mapping at the MAC layer on its immunity to PN at the physical layer. We compare the localized and distributed subcarrier mapping...
same, i.e., no near-far effects, we conclude that, for the NM SE
respectively. The quantities
compensation. Since no pilots are inserted in the data SC-
both users have the same channel
addition to the off-diagonal entries accounting for the int ra-
$N$-th user as follows
where $P_{C,t}^l \triangleq P_r^l P_t^l$ is the composite transmit-receive PN
matrix given by
$P_{C,t}^l = \text{diag} \left( \exp(j\phi_{0}^{C,t}), \exp(j\phi_{1}^{C,t}), \ldots, \exp(j\phi_{N-1}^{C,t}) \right)$  
(47)
$\phi_{n}^{C,t} = \phi_{n}^{t} + \phi_{n}^{r} = \phi_{n-1}^{t} + \phi_{n-1}^{r} + e_{n-1}^{t} + e_{n}^{r}$  
(48)
Since $e_{n}^{t} \sim \mathcal{N}(0, \frac{2\pi \beta^{2}}{N_{f}}} )$ and $e_{n}^{r} \sim \mathcal{N}(0, \frac{2\pi \beta}{N_{f}} )$, then $e_{n}^{t} + e_{n}^{r} \sim \mathcal{N}(0, \frac{2\pi \beta^{2} + 2\pi \beta}{N_{f}} )$. With the approximation in (46), the transmit and receive PN effects are lumped together and treated as transmit PN only whose effective 3-dB linewidth is $\beta_{C,t} = \beta_{t} + \beta_{r}$. We exchanged the channel matrix with the receive rather than transmit PN matrix for two reasons. First, the former yields a more accurate approximation because the receive PN process has a narrower spectral width and, hence, better satisfies the condition above. Second, lumping both transmit and receive PN at the transmitter enables us to move the channel equalizer out of the iterative channel decoding and PN compensation loop as shown in Section V-B which significantly reduces the hardware complexity of our proposed approach. Denoting by $k$ the index of the user of interest, we obtain its decision statistic by applying the $N$-point FFT to $y^{t}$ and selecting its assigned $M$ subcarriers by applying the matrix $S_{k}^{H}$. Hence, we write the decision statistic of the $k$-th user as follows
\[
Y_{k}^{t} = S_{k}^{H} F_{N} X^{t} = S_{k}^{H} \overline{H}_{k}^{t} P_{t}^{t} S_{k} F_{M} X_{k}^{t} + Z_{k}^{t}  \quad (49) \]
where $Z_{k}^{t} = S_{k}^{H} F_{N} X^{t} + S_{k}^{H} F_{N} \sum_{l \neq k} H_{l}^{t} P_{l}^{t} g_{l}^{t}$ represents the noise plus the IUI caused by the PN-induced leakage between users’ subcarriers. Furthermore, $\overline{H}_{k}^{t} = F_{N} H_{k}^{t} F_{N}^{H}$ is the $N \times N$ diagonal FD channel matrix and $P_{C,k}^{t} = F_{N} P_{C,k}^{t} F_{N}^{H}$ is the $N \times N$ circulant PN matrix. The ICI is induced by the matrix $P_{C,k}^{t}$ due to its non-diagonal (circulant) structure. However, for practical PN levels, $P_{C,k}^{t}$ is a banded matrix with few significant diagonals resulting in ICI only from few neighboring subcarriers. Since $S_{k}^{H} S_{k} = I_{M}$, we write
\[
Y_{k}^{t} = S_{k}^{H} S_{k} S_{k}^{H} \overline{H}_{k}^{t} P_{t}^{t} S_{k} F_{M} X_{k}^{t} + Z_{k}^{t}  \quad (50) \]
where $S_k S_k^H$ is a diagonal matrix whose diagonal entries are zeros except for those whose indices lie in $J_k$. Since both $H_k$ and $S_k S_k^H$ are diagonal, we can interchange them as follows

$$Y_k^t = S_k^H H_k S_k \tilde{P}_C k S_k F_M x_k^t + Z_k^t \equiv H_k \tilde{P}_C k S_k F_M x_k^t + Z_k^t$$

(51)

where $H_k$ and $\tilde{P}_C k$ are sub-matrices of $H_k$ and $\tilde{P}_C k$, respectively, whose row and column indices lie in $J_k$. Note that $H_k$ is an $m \times m$ diagonal matrix while $\tilde{P}_C k$ is an $m \times m$ Toeplitz [20] matrix. The next detection step is to apply FD equalization to equalize the channel effects in the matrix $H_k$. Applying the MMSE equalizer, we get

$$Y_{k, eq}^t = E_k Y_k^t$$

(52)

where $E_k = \left( \frac{H_k}{H_k + \frac{1}{\gamma}} \right)^{-1} \left( \frac{H_k}{H_k} \right)^H \equiv S_k D_k S_k$. Note that $H_k$ is a diagonal matrix and so is $E_k$. Next, we apply the $M$-point inverse FFT to go back to TD to detect the $k$-th user's data. Meanwhile, we remove the bias factor, $\alpha_k$, induced by the MMSE equalization to get

$$Y_{k, TD}^t = \frac{1}{\alpha_k} F_M^H Y_{k, eq}^t = F_M^H \tilde{P}_C k F_M x_k^t + Z_{k, TD}^t$$

(53)

where $\tilde{P}_C k \triangleq F_M^H \tilde{P}_C k F_M$. The vector $Z_{k, TD}^t$ contains the noise and residual ICI due to MMSE equalization, and $\alpha_k$ is the average of the diagonal entries of the diagonal matrix $E_k H_k$. Exploiting the fact that Toeplitz matrices of large sizes can be approximated as circulant matrices [20], we approximate the Toeplitz matrix $\tilde{P}_C k$ by a circulant one. Hence, we approximate the matrix $\tilde{P}_C k$ by a diagonal matrix since circulant matrices are diagonalized by the FFT matrix. Note that $\tilde{P}_C k$ is a sub-matrix of $P_C k = F_N P_C k F_N^H$ where $P_C k$ is a diagonal matrix whose diagonal entries are complex exponentials of unit magnitude as in (4). Hence, the magnitudes of the diagonal entries of $P_C k$ are also approximately unity. Intuitively, the matrix $P_C k$ can be viewed as a PN matrix directly multiplying the complex data in $x_k^t$ as in (53). In SC-FDMA, the complex data in $x_k^t$ are in TD; hence, adding the PN samples directly to their phases is intuitively appealing. Finally, $Y_{k, TD}^t$ in (53) is equivalently re-written as follows

$$Y_{k, TD}^t = X_k^t p_k^t + Z_{k, TD}^t$$

(54)

where $X_k^t$ is an $M \times 1$ diagonal matrix whose diagonal entries are given by the vector $x_k^t$ while $p_k^t$ is an $M \times 1$ vector containing the diagonal entries of the matrix $P_C k$.

B. Iterative Processing

We describe our proposed iterative channel decoding and PN compensation approach depicted in Fig. 2. For the $t$-th SC-FDMA symbol, the following procedure is executed

**Initialization:** Initialize the estimate of the diagonal PN matrix $P_C k$ with the $M \times M$ diagonal matrix $P_C k^{0}$, and initialize the iteration counter with $i = 0$.

**The $i$th iteration:**

1. Using $\hat{P}_C k$ and based on (53), we compensate for PN in $Y_{k, TD}^t$ and obtain

$$Y_{k, comp}^t = \left( \hat{P}_C k \right)^{-1} Y_{k, TD}^t \simeq x_k^t + \left( \hat{P}_C k \right)^{-1} Z_{k, TD}^t$$

(55)

where $\left( \hat{P}_C k \right)^{-1}$ is easily calculated since $\hat{P}_C k$ is an $M \times M$ diagonal matrix.

2. Compute the Log-likelihood-ratios (LLRs) of the code bits corresponding to the $m$-th entry of $Y_{k, comp}^t$, $0 \leq m \leq M - 1$, as follows

$$L_{m}^{2, i} = \log \left( \frac{\sum_{\theta \in S_1} \exp \left( \frac{1}{\sigma_m^2} \left| Y_{k, comp}^t (m) - \theta \right|^2 \right)}{\sum_{\theta \in S_0} \exp \left( \frac{1}{\sigma_m^2} \left| Y_{k, comp}^t (m) - \theta \right|^2 \right)} \right)$$

(56)

where $0 \leq q \leq Q - 1$. $Q$ is the number of code bits represented by any constellation symbol, $2^Q$ is the constellation size. Furthermore, $\sigma_m^2 = \mathbb{E} \left[ |Z_{k, comp}^t (m)|^2 \right]$ and $S_1$ and $S_0$ are the sets of signal constellation symbols where the $q$-th bit is equal to 1 and 0, respectively. The LLR in (56) can be approximated by

$$L_{m}^{2, i} \simeq \frac{1}{\sigma_m^2} \left( \min_{\theta \in S_1} \left| Y_{k, comp}^t (m) - \theta \right|^2 \right) - \min_{\theta \in S_0} \left| Y_{k, comp}^t (m) - \theta \right|^2$$

(57)

3. De-interleave the LLRs obtained in Step 2 and pass them to the channel decoder which updates them using the channel code structure. Then, de-interleave the updated LLRs and denote them by $\{ \hat{L}_{m}^{2, i} \}$.

4. Using $\{ \hat{L}_{m}^{2, i} \}$, compute soft estimates for the signal constellation symbols in $x_k^t$ as follows

$$\hat{x}_k^t (m) = \mathbb{E} \left[ x_k^t (m) \right] = \sum_{b=1}^{Q} \theta_b P \left( x_b^t (m) = \theta_b \right)$$

(58)

where $\theta_b$ is a summation variable that takes on all values of the constellation points and $P \left( x_b^t (m) = \theta_b \right)$ is the

![Fig. 2. Iterative Channel Decoding and PN Compensation](image-url)
probability that \( x^t_k(m) = \theta_b \) calculated as follows
\[
P \left( x^t_k(m) = \theta_b \right) = \prod_{q=0}^{Q-1} P \left( c^q_m = \theta_b \right)
\]
\[
= \prod_{q=0}^{Q-1} \frac{1}{2} \left( 1 + \left( 2c^q_k - 1 \right) \tanh \left( \frac{I^q_c}{2} \right) \right)
\]
where \( \tanh(\cdot) \) is the hyperbolic tangent function and \( c^q_m, c^q_k \in \{0,1\} \) represent the \( q \)-th code bit corresponding to the signal constellation symbols \( x^t_k(m) \) and \( \theta_b \), respectively. The second equality in (59) is obtained by observing that \( I^q_c \) can be written as \( \log p(\hat{c}^q_1 = 1) / p(\hat{c}^q_0 = 0) \).

5) An updated estimate of the PN vector \( \hat{p}^i_k \) can be obtained by a simple point-wise division of the vector \( Y_{k,TD}^t \) by the vector \( \hat{x}^{t,i}_k \) obtained in (58). However, this approach does not exploit the low-pass nature of the PN process rendering the PN samples in \( \hat{p}^i_k \) correlated. Two possible approaches can be followed to exploit the correlation between the PN samples. We call the first approach "FD-processing" where the PN vector \( \hat{p}^i_k \) is written in terms of its FD transform \( \hat{p}^i_k \) as follows
\[
\hat{p}^i_k = F^H_{M,LP} \hat{P}^i_k
\]
where \( \hat{P}^i_k \) contains \( D << M \) significant entries of \( \hat{p}^i_k \) and \( F^H_{M,LP} \) is an \( M \times D \) tall matrix containing the columns of \( F^H_{M,LP} \) corresponding to the significant entries of \( \hat{P}^i_k \). Next, the linear least-squares estimate (LLSE) of \( \hat{P}^i_k \) is obtained as follows
\[
\hat{P}^i_{k,LP} = \left( A^H A \right)^{-1} A^H Y_{k,TD}^t
\]
where \( A = \hat{X}^{t,i}_k F^H_{M,LP} \) and \( \hat{X}^{t,i}_k \) is a diagonal matrix whose diagonal entries are given by the estimate symbols \( \{ \hat{x}^{t,i}_k(m) \} \) obtained in (58). The size of the matrix \( A^H A \) is \( D \times D \); hence, its inverse can be obtained easily even if \( D << M \) (typically, \( D = 3 \)). Then, an updated estimate of the PN vector \( \hat{p}^i_k \) is obtained as follows
\[
\hat{p}^i_{k} = F^H_{M,LP} \hat{P}^i_{k,LP}
\]

6) Construct the \( M \times M \) diagonal matrix \( \hat{P}^i_{C,k} \) whose diagonal entries are given by the PN vector estimate \( \hat{p}^i_k \) calculated in step 5 using either the "FD-processing" approach or the "sub-block processing" approach.

7) Set \( i = i+1 \) and check the stopping criterion. If met, exit the algorithm, else go to Step 1. Several stopping criteria can be used, e.g., the maximum number of iterations or the cyclic-redundancy check.

We emphasize that modeling both transmit and receive PN as transmit-only PN moved the equalization step outside the iterations which significantly reduced the overall complexity.

C. Initialization

In this section, we investigate the PN initialization needed for the proposed iterative approach in Section V-B. Specifically, we show how to choose the diagonal matrix \( \hat{P}^0_{C,k} \). If pilots were inserted in SC-FDMA symbols, they would be used to estimate the CPE for the initialization process as done in [5]–[7], [22] for OFDM systems. The CPE is the average of the PN samples perturbing the SC-FDMA symbol. However, the model adopted in this paper assumes no pilots within the data symbols. To overcome this problem, we exploit the correlation between PN samples perturbing consecutive SC-FDMA symbols. Hence, we choose the PN initialization of the \( t \)-th SC-FDMA symbol to be a function of the estimated PN in the \( (t-1) \)-th SC-FDMA symbol. Two choices are proposed in this paper. First, we choose the average of the PN samples estimated in the \( (t-1) \)-th SC-FDMA symbol to initialize the PN of the \( t \)-th SC-FDMA symbol as follows
\[
\hat{P}^t_{C,k} = \left( \frac{1}{M} \hat{P}^{t-1}_{C,k} \right) I_M
\]
where \( I_M \) is the \( M \times 1 \) all-ones vector and \( \hat{P}^{t-1}_{C,k} \) is the PN vector estimate obtained at the last iteration of the \( (t-1) \)-th SC-FDMA symbol detection. The second choice is the last sample of the estimated PN vector as follows
\[
\hat{P}^t_{C,k} = \hat{p}^{t-1}_{k}(M) I_M
\]

The PN of the first SC-FDMA symbol is initialized with \( \hat{P}^0_{C,k} = I_M, t = 0 \). This is a reasonable choice since a training symbol usually precedes the data symbols in all frames, as in the LTE standard [9], for timing and phase synchronization.
Fig. 3. Comparison between analytical (solid lines) and simulated (dashed lines) NMSE with localized and distributed mapping schemes under various levels of CFO and PN

Fig. 4. Comparison between analytical (solid lines) and simulated (dashed lines) NMSE with joint transmit-receive PN and CFO for \( N = 128 \)

**VI. SIMULATION RESULTS**

In this section, we present numerical results for our derived NMSE expression and our proposed PN compensation approach. Unless otherwise is stated, we model SC-FDMA systems with \( N = 512, L_p = 32, f_{\text{sub}} = 15\text{kHz}, M = 96, \) and the total number of guard subcarriers on both sides is 212 as specified in the LTE uplink standard [9] for 5 MHz bandwidth. We use 16-QAM modulation for all simulations except for Fig. 8 where we use QPSK. We also use rate-1/2 convolutional coding with MAP decoding.

**A. NMSE Expressions Verification**

In this subsection, we numerically verify the derived NMSE expression by Monte-Carlo simulations. For all users, we use the exponentially-decaying multi-path channel model with 6 uncorrelated paths and 1 dB decaying factor per path. We use the localized subcarrier mapping scheme shown in Fig. 1a. We assume that all users have the same received SNR level and transmit PN and CFO parameters, i.e., \( \gamma_l = \gamma, \beta_l = \beta, \) and \( \Delta f_l = \Delta f, \forall l \). In Fig. 3, we verify the observation made in Section IV-B about the effect of subcarrier mapping on the NMSE. We compare the analytical and simulated NMSE of user \( U_1 \) with both distributed and localized subcarrier mapping schemes under various CFO and PN levels. In addition to the accuracy of the analytical NMSE expression, we notice that the localized subcarrier mapping scheme is more immune to CFO and PN than the distributed subcarrier mapping scheme as discussed in Section IV-B. In Fig. 4, we numerically verify the NMSE expression for small \( N = 128 \) to validate our approximation of Toeplitz matrices with large size as circulant matrices. The accuracy of the NMSE expression derived in Section III is evident even for SNR levels as low as 8 dB and for small FFT sizes as small as \( N = 128 \). Note that the NMSEs of both mapping schemes are identical in the case of no CFO/PN because there is neither IUI nor ICI. We also observe that the NMSE gap between both schemes increases as the CFO/PN level increases. The derived NMSE expression can be used to investigate the near-far effect by allowing different receive SNR levels (\( \gamma_l \)) for different users.

**B. Performance Evaluation of the Proposed PN Compensation Algorithm**

In this subsection, we simulate the performance of our proposed iterative compensation algorithm. The data frame consists of 6 data SC-FDMA symbols over which the channel is quasi-static as in the LTE uplink subframe [9]. The transmit and receive PN levels are set to \( \beta_l = 110 \text{ Hz} \\forall l, \beta_r = 12 \text{ Hz} \) which correspond to oscillator PSDs of \(-47.6 \text{ dBc/Hz}\) and \(-57.2 \text{ dBc/Hz}\), respectively, at 1 kHz frequency offset from the oscillator carrier frequency. We choose \( \beta_r \) to be greater than \( \beta_r = 12 \) since users’ oscillators are less accurate than the BST’s oscillator. We use the localized subcarrier mapping scheme and assume enough guard subcarriers between users; hence, we ignore the PN-induced leakage between users’ edge subcarriers. In Fig. 5, we show the frame-error-rate (FER) performance of five iterations of our “sub-block processing-based” approach without initialization, i.e., \( \hat{P}_{C,k}^{t,0} = \mathbf{I}_M, \forall t, \) and with the two initializations proposed in (65) and (66). Clearly, both proposed initializations show significant performance improvement over the case without initialization [23]. Furthermore, the initialization in (66) outperforms that in (65) because the last PN sample of the \( (t-1)\text{th} \) SC-FDMA symbol is the closest one to the PN samples of the \( t\)-th SC-FDMA symbol. Linear interpolation is used and the number of sub-blocks \( C \) is varied over iterations as follows. In the early iterations, the SC-FDMA symbol is divided into few sub-blocks since the symbols’ estimates are still inaccurate and, hence, more averaging is needed. In later iterations with improved data symbols estimates, the SC-FDMA symbol is divided into more sub-blocks to improve the accuracy of the PN interpolation. In the 1st and 2nd iterations, \( C \) is chosen to be 1 and 2, respectively. Then, we increment \( C \) by 2 for each subsequent iteration. Similarly, for the “FD-processing” approach, the parameter \( D \) is chosen to increase over iterations where \( D = 1 \) in the 1st iteration and then incremented by 2 for each subsequent iteration. In Figs. 6 and 7, we compare the performances of our “FD-processing-based” and “sub-block processing-based” iterative approaches for the SCM [24] suburban and urban macro channel models, respectively, using the initialization in (66). The latter is shown to outperform the
former which is intuitively appealing since "FD-processing" approximates the non-periodic PN process by a periodic one by keeping only few significant FFT coefficients. This leads to high estimation errors especially at both edges of the PN block [7], [25]. The impact of channel estimation errors is shown also in Fig. 7 where the pilots preceding each data frame are used to compute the standard LLSE of the channel frequency response at each subcarrier. The pilots are generated using the Zadoff-Chu sequences as in the LTE standard [9]. For channel estimation, we assume that the training sequence is being used to mitigate PN impacting the training sequence using any of the existing techniques in the literature such as [4], [26], [27] followed by PN-free channel estimation. Note that the estimated PN samples during the training sequence are different from PN samples during the consecutive data symbols, hence they cannot be used for data detection. Our proposed PN compensation algorithms experience insignificant SNR loss of 0.3 dB at FER=10^{-2} due to channel estimation errors; hence, they are robust under channel estimation errors. For heavy multimedia traffic, the error floors experienced by our approaches (due to channel estimation errors) might be higher than the required FER of around 10^{-3}. In this case, the link adaptation (LA) protocol is initiated to switch to a lower constellation size (e.g. switch from 16-QAM to QPSK) which achieves lower FER (about one order-of-magnitude lower) without having error floors as shown in Fig. 8. In this figure, we compare the performances of our proposed PN compensation algorithms with the MAP-based one proposed in [11] for QPSK with known and estimated channels. We observe that our proposed algorithms and the MAP-based one almost achieve the same performance with known and estimated channels. However, our algorithms have the advantage that they are applicable to any modulation scheme and do not require the optimization of any parameters unlike the MAP algorithm in [11] where careful choices of some design parameters are required to satisfy the stability conditions. Furthermore, our sub-block-based algorithm requires less complexity than the MAP algorithm where the former and later require, respectively, O(8M) and O(13M) real multiplications per iteration. Also, our algorithm does not involve the evaluation of several hyperbolic functions as in [11]. The performances of our proposed algorithms versus the PN level are shown in Fig. 9. 

![Fig. 5. Performance of “sub-block processing” using no initialization (dashed-dotted), initialization in (65) (dashed), and initialization in (66) (solid) for the AWGN channel.](image1)

![Fig. 6. Performance of “FD-processing” (dashed) and “sub-block processing” (solid) with the initialization in (66) for the SCM macro suburban channel.](image2)

![Fig. 7. Performance of various algorithms for 16-QAM using known (solid) and estimated (dashed) channel for SCM macro urban channel.](image3)

![Fig. 8. Comparison with MAP-based algorithm in [11] for QPSK using known (solid) and estimated (dashed) channel for SCM macro urban channel.](image4)
We derived a new NMSE expression for SC-FDMA uplink transmissions under CFO and joint transmit-receive PN taking into account MMSE channel equalization effects. The resulting expression is a function of the CFO and PN levels, the active users’ received SNR levels and subcarrier assignments, and the users’ channel delay profiles. Based on the derived NMSE expression, we compared the localized and distributed subcarrier mapping schemes from the viewpoint of their immunity to CFO and PN. We proposed an iterative reduced-complexity algorithm for joint decoding and PN compensation in SC-FDMA systems with the equalizer moved out of the loop to reduce complexity. As in the LTE uplink, no pilots are multiplexed with data subcarriers to keep the low-PAPR feature of SC-FDMA systems. Hence, our proposed PN compensation algorithm does not use pilots for PN tracking. Instead, we use the decoded data iteratively for PN estimation and compensation. Furthermore, the PN low-pass nature is exploited without assuming a-priori knowledge about the exact PN model to avoid model mismatch problems. Significant performance gains are demonstrated by the simulation results.

VII. CONCLUSION

We derived a new NMSE expression for SC-FDMA uplink transmissions under CFO and joint transmit-receive PN taking into account MMSE channel equalization effects. The resulting expression is a function of the CFO and PN levels, the active users’ received SNR levels and subcarrier assignments, and the users’ channel delay profiles. Based on the derived NMSE expression, we compared the localized and distributed subcarrier mapping schemes from the viewpoint of their immunity to CFO and PN. We proposed an iterative reduced-complexity algorithm for joint decoding and PN compensation in SC-FDMA systems with the equalizer moved out of the loop to reduce complexity. As in the LTE uplink, no pilots are multiplexed with data subcarriers to keep the low-PAPR feature of SC-FDMA systems. Hence, our proposed PN compensation algorithm does not use pilots for PN tracking. Instead, we use the decoded data iteratively for PN estimation and compensation. Furthermore, the PN low-pass nature is exploited without assuming a-priori knowledge about the exact PN model to avoid model mismatch problems. Significant performance gains are demonstrated by the simulation results.

REFERENCES