Stat 6329 Syllabus

Stat 6329.001
Applied Probability and Stochastic Processess
Course Information

Course number/section	Stat 6329-001
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Required text	Introduction to Probability Models, 10^{th} or 11^{th} ed.
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Suggested course materials	None

Tentative Schedule

Topics	Chapters
Review of probability and random variables	1.1-1.6; 2.1-2.3
Expectation	2.4
Joint distributions and independence	2.5-2.8
Conditioning and conditional expectation	3.1-3.7
Markov chains	4.1-6
Poisson and related processes	5.1-5.4
Renewal processes	7.1-7.3
Introduction to queuing models	8.1-3
Introduction to Brownian motion	10.1-10.6

Grading Policy

Final course grade will be based on 3 exams and homework. You may bring one sheet of notebook paper with notes for the exams.

Student Learning Objectives

1. Understand the fundamentals of probability theory and random variables
2. Understand the basic probability tools for stochastic processes: conditioinal probability and conditional expectation

3. Understand the basic properties and application of special classes of stochastic processes including Markov chains and Poisson and related processes

The complete syllabus is available here:
`http://www.utdallas.edu/~ammann/stat6329_syllabus.pdf`

Class Notes

Review of Probability and Random Variables

Probability provides a mathematical model for an *Experiment*: a process whose outcome is uncertain. A probability model contains three components.

1. Ω, the *Sample space* defined to be the set of all possible outcomes of the experiment
2. $\mathcal{A}$, a collection of events (subsets of Ω) whose probabilities we wish to model. For reasons of mathematical consistency this collection must contain Ω, complements of any set in $\mathcal{A}$ (which implies $\emptyset \in \mathcal{A}$), along with countable unions of sets in $\mathcal{A}$. We refer to such collections as sigma-algebras and events in such a collection are referred to as measureable sets. The collection of Borel subsets of $\Re$ is an example. If Ω is finite or uncountable infinite, then $\mathcal{A}$ can be the set of all subsets of Ω. However, if Ω is uncountable, then we need these restrictions.
3. A probability function $P : \mathcal{A} \to \Re$ that satisfies
   (a) $0 \leq P(A) \leq 1, \forall A \in \mathcal{A}$
   (b) $P(\Omega) = 1$
   (c) if $\{E_i\}$ is a countable collection of disjoint events in $\mathcal{A}$, then
   $$P\left(\cup E_i\right) = \sum_i P(E_i).$$

Note that these components, in particular the probability function, are defined in terms of the underlying experiment.

Properties

1. $P(\emptyset) = 0$. **Proof**: let

$$E_i = \emptyset,\ i \geq 1.$$

Then $\{E_i\}$ are disjoint and

$$\cup E_i = \emptyset.$$

Hence,

$$P(\emptyset) = P\left(\cup E_i\right) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{\infty} P(\emptyset).$$

The only real number that satisfies this equation is $P(\emptyset) = 0$.

2. If $E_1, \cdots E_n$ is a finite collection of disjoint events in $\mathcal{A}$, then

$$P\left(\cup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i).$$

Proof: let $E_k = \emptyset$, $k \geq n+1$. Then $\{E_i,\ i \geq 1\}$ is a countable collection of disjoint events in $\mathcal{A}$ and

$$\cup_{i=1}^{n} E_i = \cup_{i=1}^{\infty} E_i$$

Hence,

$$P\left(\cup_{i=1}^{n} E_i\right) = P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) = \sum_{i=1}^{n} P(E_i).$$

3. If $A, B \in \mathcal{A}$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof: write

$$A \cup B = A \cup \left(B \cap A^C\right)$$

and note that this is a disjoint union. Therefore

$$P(A \cup B) = P(A) + P(B \cap A^C).$$

Similarly,

$$P(B) = P(A \cap B) + P(A^C \cap B).$$

Hence,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Note that this implies the corollary: if A,B are measureable sets with $A \subset B$, then $P(A) \leq P(B)$. So probability is a monotone function. More generally, let $E_1, \cdots E_n$ be an arbitrary collection of measureable sets. Then

$$P\left(\cup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < \cdots < i_r \leq n} P\left(E_{i_1} \cap \cdots \cap E_{i_r}\right).$$

Proof: by induction. This is referred to as the *inclusion-exclusion* identity. For example,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

4. Experiments with equally likely outcomes. Suppose Ω contains n outcomes, $O_1, \cdots O_n$ and each outcome has the same probability,

$$P(O_j) = p,\ 1 \leq j \leq n.$$

Then

$$1 = P(\Omega) = P(\cup_i O_i) = \sum_{i=1}^{n} P(O_j) = np,$$

which implies that $p = 1/n$. Furthermore, since any event in $\mathcal{A}$ can be expressed as a union of individual outcomes, then

$$P(E) = \frac{\#\{E\}}{n}.$$

Example. What is the probability at least two people in a group have the same birthdate? Ignore leap-years and assume that 365 possible dates in a year are equally likely. Suppose the group size is $n < 365$ and no twins, triplets, etc., are in the group. Then the total number of outcomes for the n birthdates is 365^n. The number of outcomes with different dates is

$$(365)(364)\cdots(365-n+1),$$

so

$$P(\text{no match}) = \frac{365!}{(365-n)!365^n} = \begin{cases} 0.49, & n = 23 \\ 0.29, & n = 30 \\ 0.03, & n = 50 \end{cases}$$

Conditional Probability

We often use the language of percentages when describing probabilities. For example, suppose a large company has 10 positions to fill and has received 100 qualified applicants for those positions. Suppose a review of those applicants gave the following:

	M	F
Hired	8	2
not	52	38

Assume a probability model in which each qualified applicant is equally likely to be hired. Then we can say
1) 10% of qualified applicants were hired, $P(\text{Hired}) = 10/100$
2) 60% of qualified applicants were male, $P(M) = 60/100$
3) 8% of qualified applicants were male and hired, $P(M \cap \text{Hired}) = 8/100$
But what about the statements:
4) 80% of those hired were male, 8/10
5) 5% of qualified female applicants were hired, 2/40.

The last two statements require a different type of probability. Note the English structure of the first statement:
(10%) (of qualified applicants) (were hired)
(subject) (adjective phrase modifying subject) (verb plus object)
Note that the adjective phrase defines the reference group for the percentage and so specifies the denominator. The verb plus object defines the outcomes contained in the event. The translation into the language of probability is then

$$0.1 = \frac{\#\{\text{were hired}\}}{\#\{\text{qualified applicants}\}} = \frac{10}{100}.$$

This is ordinary probability because the reference group is the entire sample space. Now consider statement 4) above.
(80%) (of those hired) (were male)
The reference group is the set of those who were hired. Since this is not the entire sample space, then this is not an ordinary probability. Also, the requirement (were male) must be counted from within the reference group. Therefore, we can translate this statement as:

$$0.8 = \frac{\#\{\text{were male and hired}\}}{\#\{\text{were hired}\}} = \frac{P(M \cap H)}{P(H)}.$$

Similarly, 5) is equivalent to

$$0.05 = \frac{P(F \cap H)}{P(F)}.$$

We refer to these as conditional probabilities, defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B) > 0.$$

Note that conditional probability is only defined for conditions with positive probabilities. The translations of 4) and 5) are:

$$\begin{aligned} P(\text{male}|\text{hired}) &= \frac{8/100}{10/100} = 0.8 \\ P(\text{hired}|\text{female}) &= \frac{2/100}{40/100} = 0.05. \end{aligned}$$

Note that the reference group comes after the seperator — in these conditional probabilities.
Properties of Conditional Probability

1. $$P(AB) = P(A|B)P(B) = P(B|A)P(A).$$
   Note that if $P(A) = 0$ or $P(B) = 0$ then $P(AB) = 0$ from the monotonicity property of probability.

2. Let B denote an event with $P(B) > 0$ and define a function P_B on $\mathcal{A}$ by:
   $$P_B(E) = P(E|B),\ E \in \mathcal{A}.$$
   The P_B satisfies the axioms of probability and therefore it also satisfies all the properties derived from those axioms.

3. Let $\{B_i\}$ denote a finite or countably infinite *partition* of Ω, that is, these events are disjoint and their union is Ω. Assume without loss of generality that $P(B_i) > 0,\ \forall i$. Then
   $$P(A) = P\left(\cup_i (AB_i)\right) = \sum_i P(AB_i) = \sum_i P(A|B_i)P(B_i).$$

This is referred to as the *Theorem of Total Probability.* This can be extended to give Bayes Theorem: if $P(A) > 0$ then

$$P(B_k|A) = \frac{P(AB_k)}{P(A)} = \frac{P(A|B_k)P(B_k)}{\sum_i P(A|B_i)P(B_i)}.$$

In this context $P(B_i)$ is referred to as a prior probability and $P(B_k|A)$ is referred to as a posterior probability. The posterior probability of B_k represents an update to the likelihood that event B_k will occur given that we have observed an outcome in A.

Example. Suppose a particular type of cancer with a high mortality rate is very costly and/or risky to diagnose with standard tools such as surgical biopsy. Suppose also that the set of symptoms experienced by patients with this cancer is relatively common so that only 5% of all patients with those symptoms actually have this type of cancer. Then this cancer is just one possible diagnosis for such patients, but it may not the most likely diagnosis. Now suppose a relatively inexpensive, low risk screening test has been developed to help diagnose this type of cancer for the target population, the set of patients who exhibit that set of symptoms. Properties of the screening test are obtained by applying it to patients in the target population, some of whom are known to have this type of cancer, the rest are known not to have this cancer. Suppose the following results were obtained: 98% of those in the target population known to have this cancer get a positive response to the screening test and 95% of those in the target population known not to have this cancer get a negative response. Is this a good screening test? We will answer this question from the perspective of a doctor who has a patient identified as belonging to the target population. Suppose this patient gets a positive response to the screening test. What does the doctor tell the patient?

First we construct a probability model for the experiment in which a patient is randomly selected from the target population. Then the translations of the results into this probability model are:

$$\begin{aligned} P(\text{has cancer}) &= 0.05, \\ P(\text{Positive}|\text{has cancer}) &= 0.98 \\ P(\text{Negative}|\text{not cancer}) &= 0.95 \end{aligned}$$

This gives

$$\begin{aligned} P(\text{Positive} \cap \text{has cancer}) &= P(\text{Positive}|\text{has cancer})P(\text{has cancer}) \\ &= (0.98)(0.05) = 0.0490 \\ P(\text{Negative} \cap \text{not cancer}) &= P(\text{Negative}|\text{not cancer})P(\text{not cancer}) \\ &= (0.95)(0.95) = 0.9025 \end{aligned}$$

We can put these into a probability table:

	has cancer	not cancer	total
Positive	.0490		
Negative		.9025	
total	.0500		1.0000

Note that this table only contains ordinary probabilities, not conditional probabilities. The blank cells can be filled in using the theorem of total probability:

	has cancer	not cancer	total
Positive	.0490	.0475	.0965
Negative	.0010	.9025	.9035
total	.0500	.9500	1.0000

We can obtain the answer to the question from this table:

$$P(\text{has cancer}|\text{Positive}) = \frac{.0490}{.0965} = 0.508,$$

not much better than a coin flip! How can that be a good test? But note that before the screening test result is known, the probability the patient has this type of cancer is 5%; after a positive test result, this probability increases to 50.8% from 5%. This increased risk would warrant applying the more costly definitive diagnostic tool to the patient. On the other hand, if the screening test is negative, then

$$P(\text{has cancer}|\text{Negative}) = \frac{.0010}{.9035} = 0.0011.$$

In this case the doctor would test for other diagnoses. Of course, if it turns out that none of these other diagnoses is effective, the doctor likely would then apply the definitive diagnostic test for this type of cancer given its high mortality rate.

Example: the Monte Hall problem. This problem is named after the host of a TV game show called *Let's Make a Deal*. A contestant is presented with 3 doors behind one of which is a valuable prize but the other two doors contain nothing of value. Monte asks the contestant to choose a door and the contestant will win whatever is behind the door. However, before opening the selected door, Monte opens one of the other doors he knows does not contain the valuable prize and offers the contestant the opportunity to switch his or her choice to the remaining unopened door. Should the contestant switch or stay with the original choice?

This problem first appeared in 1975 in the *American Statistician*. Quite a few people who answered based on their intuition and said the remaining unopened doors are equally likely to have the valuable prize got it wrong. It is interesting to note that the audience members who were fans of the show got it right - they always encouraged the contestant to switch! To see why, we will construct a probability model for this problem. Assume the prize is equally likely to be behind any of the doors:

$$P(A) = P(B) = P(C) = 1/3.$$

Suppose the contestant chooses door A. If the prize is behind door A, assume Monte mentally flips a coin to select which of the other doors to open, but if the prize is not behind A then

Monte must open the door he knows does not have the prize. Suppose Monte opens door C and then asks the contestant if he or she would like to switch from A to B. We need to find the conditional probability,

$$P(A|\text{opens C}) = \frac{P(A \cap \text{opens C})}{P(\text{opens C})}.$$

Now,

$$P(A \cap \text{opens C}) = P(\text{opens C}|A)P(A) = (1/2)(1/3) = 1/6.$$

Also,

$$\begin{aligned} P(\text{opens C}|A)P(A) &= (1/2)(1/3) = 1/6 \\ P(\text{opens C}|B)P(B) &= (1)(1/3) = 1/3 \\ P(\text{opens C}|C)P(C) &= (0)(1/3) = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} P(\text{opens C}) &= P(\text{opens C}|A)P(A) + P(\text{opens C}|B)P(B) + P(\text{opens C}|C)P(C) \\ &= (1/2)(1/3) + (1)(1/3) + (0)(1/3) \\ &= 1/2 \end{aligned}$$

And so,

$$\begin{aligned} P(A|\text{opens C}) &= \frac{1/6}{1/2} = \frac{1}{3} \\ P(B|\text{opens C}) &= \frac{1/3}{1/2} = \frac{2}{3} \end{aligned}$$

Switch!

Independence

If $P(A|B) = P(A)$, then knowledge that the outcome of the experiment is in B does not change the likelihood that A occurs. In this case we say that events A and B are statistically independent. Note that this would imply that

$$P(AB) = P(A|B)P(B) = P(A)P(B)$$

Since $P(A|B)$ is not defined when $P(B) = 0$, we define independence more generally by
Definition: events A and B are said to be *statistically independent* iff $P(AB) = P(A)P(B)$. This definition is equivalent to $P(A|B) = P(A)$ when $P(B) > 0$.

Random Variables

We use r.v.'s to model numeric data. A r.v. assigns a numeric value to each outcome of an experiment. Formmally, a r.v. X is a function, $X : \Omega \to \Re$ that satisfies

$$\{\omega \in \Omega : \ X(\omega) \leq a\} \in \mathcal{A}, \ \forall a \in \Re.$$

This requirement implies that events in which the random variable takes values in an interval will be measureable,

$$P(\omega \in \Omega : \ a < X(\omega) \leq b).$$

Note that ordinarily we suppress the notation that X is a function and write this probability as

$$P(a < X \leq b),$$

but we always should remember that this is just shorthand notation and X is actually a function on the sample space.

Suppose for example we toss a fair coin until a 6 appears. Assume the tosses are independent and let S denote the event that 6 appears on a toss and let F denote the complement of that event. Then the sample space of this experiment is

$$\Omega = \{S, FS, FFS, FFFS, \cdots\}.$$

Now let N denote the number of tosses required for a 6 to appear. Then $N(S) = 1$, $N(FS) = 2$, etc, and so for $n \geq 1$,

$$\begin{aligned} P(\omega \in \Omega : N(\omega) = n) &= P(N = n) \\ &= P(F \cdots FS) = [P(F)]^{n-1} P(S) \\ &= \left(\frac{5}{6}\right)^{n-1} \frac{1}{6}. \end{aligned}$$

The set of all possible values of a r.v. is called its sample space. In this example the sample space of N is the positive integers. Also,

$$\sum_{k=1}^{\infty} P(N = n) = \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{n-1} \frac{1}{6} = \frac{1/6}{1 - 5/6} = 1.$$

R.v.'s with a finite or countably infinite sample space are referred to as discrete r.v.'s. For such r.v.'s we can derive the probability of any event from its *probability mass function*, defined by

$$p(x) = P(X = x).$$

In particular,

$$P(E) = \sum_{x \in E} p(x).$$

Probability mass functions satisfy

1. $0 \leq p(x) \leq 1, \forall x \in \Re$
2. $p(x) > 0$ for at most countable values x

3. $\sum_x p(x) = 1$

Any function that satisfies these condtions is the p.m.f of some r.v.

If the sample space of a r.v. is uncountable, then we refer to such r.v.'s as continuous r.v.'s. In this case the event

$$\{\omega \in \Omega : X(\omega) = x\}$$

leads to mathematical inconsistencies and so the p.m.f. is not defined for continuous r.v.'s. However, the function

$$F(x) = P(\omega \in \Omega : X(\omega) \leq x) = P(X \leq x)$$

is defined for all r.v.'s and probabilities of events associated with a r.v. can be derived from this function, referred to as the cumulative distribution function. This function satisfies the following conditions

1. $F(x)$ is monotone, nondecreasing, and right continuous
2. $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$

Anuy function that satisfies these conditions is the c.d.f. of some r.v. In the discrete case,

$$\begin{aligned} P(X = x) &= F(x) - \lim_{\epsilon \searrow 0} F(x - \epsilon) \\ &= F(x) - F(x-). \end{aligned}$$

Note that if $p(x) > 0$ then F has a discontinuity point at x. The conditions on cdf's imply that a cdf has at most countably infinite many points of discontinuity.

Bernoulli and related r.v.'s

The simplest non-trivial r.v. is defined on an experiment, referred to as a Bernoulli trial, with 2 possible outcomes. Label these outcomes as S,F and let $P(S) = p$. Then $P(F) = 1-p$. A Bernoulli r.v. is defined by $X(S) = 1$, $X(F) = 0$. Its p.m.f. is

$$p(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Binomial. Suppose n independent Bernoulli trials are performed each with the same success probability p. Then the sample space consists of strings of n characters, each of which is either S or F. Let N denote the number of times S appears in the string, that is, N is the number of successes among n independent Bernoulli trials. Then the sample space is $0, 1, \cdots, n$ and the p.m.f. is

$$P(N = k) = \binom{n}{k} p^k (1-p)^{n-k},\ 0 \leq k \leq n$$

This r.v. is named the Binomial r.v. because its pmf is a term in a binomial series. This r.v. provides a probability model for random sampling with replacement from a population with

two types of individuals. In practice we usually perform sampling without replacement, but as long as the sample size is small compared to the population size, the binomial distribution provides a reasonable approximation.
Geometric. Now suppose we perform a series of independent Bernoulli trials with the same success probability and continue until we observe an S. Let N denote the number of trials required to obtain S. Then the pmf of this r.v. is

$$P(N = k) = (1 - p)^{k-1}p, \ k \geq 1.$$

Note that this pmf represents terms in a geometric series. An alternotive form of the geometric r.v. is to count the number of failures before the first success. Let Y denote this r.v. Then $Y = N - 1$ and so its pmf is

$$P(Y = j) = (1 - p)^j p, \ j \geq 0.$$

A generalization of the geometric is the negative binomial r.v., the number of trials required to obtain r successes, where r is a fixed positive integer. Then the sample space for N_r is $r, r+1, \cdots$ and $\{N_r = k\}$ is the event that there are exactly $r - 1$ successes among the first $k - 1$ trials and trial k is an S. Therefore

$$P(N_r = k) = \binom{k-1}{r-1} p^r (1 - p)^{k-r}, \ k \geq r.$$

Let M_r denote the number of failures before the r^{th} success, $M_r = N_r - r$. Then

$$P(M_r = j) = P(N_r = r + j) = \binom{r+j-1}{j} p^r (1 - p)^j, \ j \geq 0.$$

Poisson. A commonly used model for count data is the Poisson r.v. with pmf

$$P(N = n) = \frac{\lambda^n}{n!} e^{-\lambda}, \ n \geq 0,$$

where λ is a positive real number. Before calculators, computation of binomial probabilities was very tedious for large n. This led to one of the first limit theorems in probability: if $p_n \to 0$ in such a way that $np_n \to \lambda > 0$, then

$$\lim_{n\to\infty} \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \frac{\lambda^n}{n!} e^{-\lambda}, \ k \geq 0.$$

Markov chain examples

Example 1. Random walk with partial reflecting barrier at 0. Let

$$\begin{aligned} P_{0,0} &= \frac{1}{2},\ P_{0,1} = \frac{1}{2},\ P_{0,i} = 0,\ i > 1, \\ P_{i,i-1} &= q,\ P_{i,i+1} = p,\ i \geq 1,\ \text{where } p + q = 1, \\ P_{i,j} &= 0,\ i \geq 1,\ j \neq i-1, i+1. \end{aligned}$$

Then TPM is

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

All states communicate so this is an aperiodic, irreducible MC. Intuitively, it seems that if $q¿p$ then states should be recurrent. To determine whether or not states are recurrent we must consider the system of equations $\mathbf{v} = \mathbf{vP}$.

$$\begin{aligned} v_0 &= \frac{1}{2}v_0 + qv_1 \ \Rightarrow\ v_1 = \frac{1}{2q}v_0 \\ v_1 &= \frac{1}{2}v_0 + qv_2 \ \Rightarrow\ v_2 = \frac{p}{q}v_1 \end{aligned}$$

The remaining equations have the form

$$v_k = pv_{k-1} + qv_{k+1},\ k \geq 2.$$

Therefore,

$$p(v_{k-1} - v_k) = q(v_k - v_{k+1}),\ k \geq 2,$$

Let

$$\Delta_k = v_{k-1} - v_k,\ k \geq 2.$$

Then for $p \neq q$, we have the recursive equations,

$$\Delta_{k+1} = \frac{p}{q}\Delta_k \ \Rightarrow\ \Delta_k = \left(\frac{p}{q}\right)^{k-2}\Delta_2,\ k \geq 2.$$

Also,

$$\Delta_2 = v_1 - v_2 = \frac{v_0}{2q}(1 - \frac{p}{q}).$$

Since

$$\sum_{k=2}^{m} \Delta_k = v_1 - v_m,\ m \geq 2,$$

then

$$\begin{aligned}
v_m &= v_1 - \sum_{k=2}^{m} \Delta_k = v_1 - \sum_{k=2}^{m} \left(\frac{p}{q}\right)^{k-2} \Delta_2 \\
&= v_1 - \frac{1-(p/q)^{m-1}}{1-p/q}\Delta_2 \\
&= \frac{v_0}{2q} - \frac{1-(p/q)^{m-1}}{1-p/q}\Delta_2 \\
&= \frac{v_0}{2q} - \frac{1-(p/q)^{m-1}}{1-p/q}\frac{v_0}{2q}(1-\frac{p}{q}) \\
&= \frac{v_0}{2q}\left(\frac{p}{q}\right)^{m-1}.
\end{aligned}$$

Therefore,

$$\sum_{m=0}^{\infty} v_m$$

is convergent when $p¡q$. In that case,

$$\sum_{m=0}^{\infty} v_m = v_0 \frac{3-4p}{2-4p}.$$

Setting

$$\pi_k = \frac{v_k}{\sum v_m}$$

gives the stationary distribution,

$$\begin{aligned}
\pi_0 &= \frac{2(1-2p)}{3-4p} \\
\pi_1 &= \frac{1-2p}{(1-p)(3-4p)} \\
\pi_k &= \frac{1-2p}{(1-p)(3-4p)}\left(\frac{p}{1-p}\right)^{k-1} \quad k \geq 2.
\end{aligned}$$

Note that the mean time to return to state 0 is

$$\frac{1}{\pi_0} = \frac{3-4p}{2(1-2p)}.$$

It can be shown that this MC is null recurrent if $p = 0.5$ and all states are transient if $p > 0.5$.

Example 2. TPM is given by:

$$P = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/6 & 1/6 \\ 0 & 1/4 & 1/2 & 1/4 & 0 \end{bmatrix}$$

States are classified as follows: *(1,2)* is a recurrent class, *3* is an absorbing state (recurrent class with one member), and states *(4,5)* are transient. Then

$$Q = \begin{bmatrix} 1/6 & 1/6 \\ 1/4 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1/3 + 0 & 1/3 \\ 0 + 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/4 & 1/2 \end{bmatrix}$$

$$I - Q = \begin{bmatrix} 5/6 & -1/6 \\ -1/4 & 1 \end{bmatrix}$$

$$S = (I - Q)^{-1} = \frac{20}{19} \begin{bmatrix} 1 & 1/6 \\ 1/4 & 5/6 \end{bmatrix} = \begin{bmatrix} 24/19 & 4/19 \\ 6/19 & 20/19 \end{bmatrix}$$

S contains mean number of visits to transient states. So if this MC starts in state 4, then mean number of visits to state 4 is 24/19, mean number of visits to state 5 is 4/19, and mean time to absorption is 28/19. Likewise, if the MC starts in state 5, then mean number of visits to state 4 is 6/19, mean number of visits to state 5 is 20/19, and mean time to absorption is 26/19. Absorption probabilities are

$$SR = \begin{bmatrix} 24/19 & 4/19 \\ 6/19 & 20/19 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 9/19 & 10/19 \\ 7/19 & 12/19 \end{bmatrix}$$

Stationary distribution for recurrent class *(1,2)* is solution to

$$[\pi_1, \pi_2] = [\pi_1, \pi_2] \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}, \quad \pi_1 + \pi_2 = 1.$$

Solution is

$$\pi_1 = \frac{2}{5}, \ \pi_2 = \frac{3}{5}.$$

Therefore, mean times between visits to *1, 2* are *5/2, 5/3*, respectively.

Example 3. Branching processes. Suppose each individual in a population is able to produce offspring and let N denote the total number of offspring by an individual. Assume N is independent of the numbers produced by other individuals and has pmf

$$p_j, \; j \geq 0.$$

Let X_0 denote the total number of individuals at time 0, referred to as generation 0, and let X_1 denote the total number of all offspring from generation 0. These offspring are referred to as generation 1. In general, let X_n denote the total number of all offspring from generation *n-1*. Since X_{n+1} depends only on X_n and not on the sizes of previous generations, then X_k, $k \geq 0$, is a MC.

Since $P_{0,0} = 1$ then *0* is an absorbing state. Since offspring from individuals are i.i.d. then

$$P_{i,0} = p_o^i.$$

This implies that if $p_0 > 0$ then there is a positive probability this MC will be absorbed into state 0 and so states $i \geq 1$ are transient. This means that the MC will either be absorbed into state 0 or go to $+\infty$.

Let $Z_i^{(n-1)}$ denote number of offspring of i^{th} individual in generation *n-1* and let μ, σ^2 denote its mean and variance,

$$\begin{aligned}
\mu &= E\left(Z_i^{(n-1)}\right) = \sum_{j=0}^{\infty} j p_j, \\
\sigma^2 &= Var\left(Z_i^{(n-1)}\right) = \sum_{j=0}^{\infty} (j-\mu)^2 p_j.
\end{aligned}$$

Suppose $X_0 = 1$ and find $E(X_n)$ and $Var(X_n)$. First note that

$$\begin{aligned}
E(X_n | X_{n-1} = r) &= E\left[\sum_{i=1}^{X_{n-1}} Z_i^{(n-1)} | X_{n-1} = r\right] \\
&= \sum_{i=1}^{r} E[Z_i^{(n-1)} | X_{n-1} = r] = r\mu.
\end{aligned}$$

Hence, $E(X_n | X_{n-1}) = \mu X_{n-1}$ and so

$$E(X_n) = E[E(X_n | X_{n-1})] = \mu E(X_{n-1}), \; n \geq 1.$$

Since $X_0 = 1$, then $E(X_n) = \mu^n$.

Similarly,

$$Var(X_n | X_{n-1} = r) = \sigma^2 X_{n-1} = Var\left(\sum_{i=1}^{r} Z_i^{(n-1)}\right) = \sigma^2 r,$$

so

$$Var(X_n|X_{n-1}) = \sigma^2 X_{n-1}.$$

Therefore,

$$\begin{aligned} Var(X_n) &= E[Var(X_n|X_{n-1})] + Var[E(X_n|X_{n-1})] \\ &= E[\sigma^2 X_{n-1}] + Var(\mu X_{n-1}) \\ &= \sigma^2 \mu^{n-1} + \mu^2 Var(X_{n-1}). \end{aligned}$$

Let $a_n = Var(X_n)$. Then

$$\begin{aligned} a_n &= \sigma^2\mu^{n-1} + \mu^2 a_{n-1} \\ &= \sigma^2\mu^{n-1} + \mu^2(\sigma^2\mu^{n-2} + \mu^2 a_{n-2}) \\ &= \sigma^2\mu^{n-1} + \sigma^2\mu^n + \mu^4 a_{n-2} \\ &= \sigma^2\mu^{n-1} + \sigma^2\mu^n + \mu^4(\sigma^2\mu^{n-3} + \mu^2 a_{n-3}) \\ &= \sigma^2\mu^{n-1} + \sigma^2\mu^n + \sigma^2\mu^{n+1} + \mu^4 a_{n-3}. \end{aligned}$$

Continuing this process gives

$$a_n = \sigma^2 \sum_{k=n-1}^{2n-2} \mu^k.$$

If $\mu = 1$, then $a_n = n\sigma^2$. If $\mu \neq 1$, then

$$a_n = \sigma^2 \frac{\mu^{n-1} - \mu^{2n-1}}{1-\mu} = \sigma^2\mu^{n-1}\frac{1-\mu^n}{1-\mu}.$$

If $\mu < 1$, then

$$\mu^n = E(X_n) = \sum_{j=1}^{\infty} jP(X_n = j) \geq \sum_{j=1}^{\infty} P(X_n = j)P(X_n \geq 1).$$

This implies that

$$\lim_{n\to\infty} P(X_n \geq 1) = 0,$$

and so,

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0) = 1.$$

It can be shown (proof is more complicated) that $\pi_0 = 1$ if $\mu = 1$. Now suppose $\mu > 1$. Then first step analysis gives

$$\pi_0 = \sum_{j=0}^{\infty} P(\text{pop. dies out}|X_1 = j)p_j.$$

Now, the population dies out iff the population generated by each offspring dies out. Since the number of offspring for an individual is independent of number of offspring for other individuals, then

$$P(\text{pop. dies out}|X_1 = j) = \pi_0^j.$$

Hence,

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j p_j.$$

This solution may not be unique, so we take π_0 to be the smallest positive solution.

Suppose for example that

$$p_0 = 0.5,\ p_1 = 0.25,\ p_2 = 0.25.$$

Then $\mu = 0.75$ and so $\pi_0 = 1$. Now suppose

$$p_0 = 0.25,\ p_1 = 0.25,\ p_2 = 0.5.$$

Then $\mu = 1.25$ and π_0 satisfies

$$\begin{aligned} \pi_0 &= 0.25 + 0.25\pi_0 + 0.5\pi_0^2 \\ &\Rightarrow 0.5\pi_0^2 - 0.75\pi_0 + 0.25 = 0 \\ &\Rightarrow 2\pi_0^2 - 3\pi_0 + 1 = 0. \end{aligned}$$

Solutions are

$$\frac{3 \pm \sqrt{9-8}}{4} = \frac{1}{2},\ 1,$$

so $\pi_0 = 0.5$.

Example 4. List replacement algorithms. A data archive has 5 large drives arranged in a stack. When a request is made for a file stored on a particular drive, that drive is retrieved from the stack, the file is extracted, and then the drive is returned to the stack but not necessarily in the same position. Suppose drive retrieval time is proportional to the position of the drive in the stack. In that case we would like to have the most frequently requested drive at the top of the stack to minimize retrieval time. So let's consider a replacement algorithm in which a requested drive is returned to the top of the stack and drives that were above it are moved down one position. We can analyze this algorithm by tracking a particular drive, say drive A, and using the following model. Selections are independent, probability of selecting A is p and the probability of selecting each of the other items is $(1-p)/4 = q$ (others are equally likely). Let X_n denote position of drive A after the n^{th} selection where position 1 denotes the top of the stack. Note that if A is at position 1, then it will stay there if it is selected, otherwise it will move down to position 2. If A is at position k, $1 < k \leq 5$, then it will move to position 1 if it is selected, it will stay at position k if an item above it is selected, and it will move to position $k+1$ if an item below it is selected. The TPM therefore is

$$P = \begin{bmatrix} p & 1-p & 0 & 0 & 0 \\ p & q & 3q & 0 & 0 \\ p & 0 & 2q & 2q & 0 \\ p & 0 & 0 & 3q & q \\ p & 0 & 0 & 0 & 1-p \end{bmatrix}$$

The stationary distribution is given by

$$\begin{aligned}
\pi_1 &= p\pi_1 + p\pi_2 + p\pi_3 + p\pi_4 + p\pi_5 = p \\
\pi_2 &= (1-p)\pi_1 + q\pi_2 \\
\pi_3 &= 3q\pi_2 + 2q\pi_3 \\
\pi_4 &= 2q\pi_3 + 3q\pi_4 \\
\pi_5 &= q\pi_4 + (1-p)\pi_5.
\end{aligned}$$

We are mainly interested in π_1, the long-term proportion of times drive A is at position 1. For this algorithm, $\pi_1 = p$.

Now consider a different replacement algorithm: the requested drive is returned to position 1 if it was at 1, otherwise it is returned to 1 position above its previous position. The TPM for this algorithm is

$$P = \begin{bmatrix} 1-q & q & 0 & 0 & 0 \\ p & 1-p-q & q & 0 & 0 \\ 0 & p & 1-p-q & q & 0 \\ 0 & 0 & p & 1-p-q & q \\ 0 & 0 & 0 & p & 1-p \end{bmatrix}$$

Stationary distribution is given by

$$\begin{aligned}
\pi_1 &= (1-q)\pi_1 + p\pi_2 \Rightarrow \pi_2 = \frac{q}{p}\pi_1 \\
\pi_2 &= q\pi_1 + (1-p-q)\pi_2 + p\pi_3 \Rightarrow \pi_3 = \frac{q^2}{p^2}\pi_1 \\
\pi_3 &= q\pi_2 + (1-p-q)\pi_3 + p\pi_4 \Rightarrow \pi_4 = \frac{q^3}{p^3}\pi_1 \\
\pi_5 &= q\pi_4 + (1-p)\pi_5 \Rightarrow \pi_5 = \frac{q^4}{p^4}\pi_1
\end{aligned}$$

Also, the equation $\sum \pi_k = 1$ gives

$$1 = \pi_1 \sum_{k=0}^{4} \left(\frac{q}{p}\right)^k.$$

If $p = q$ then

$$\pi_1 = \frac{1}{5} = \pi_2 = \pi_3 = \pi_4 = \pi_5.$$

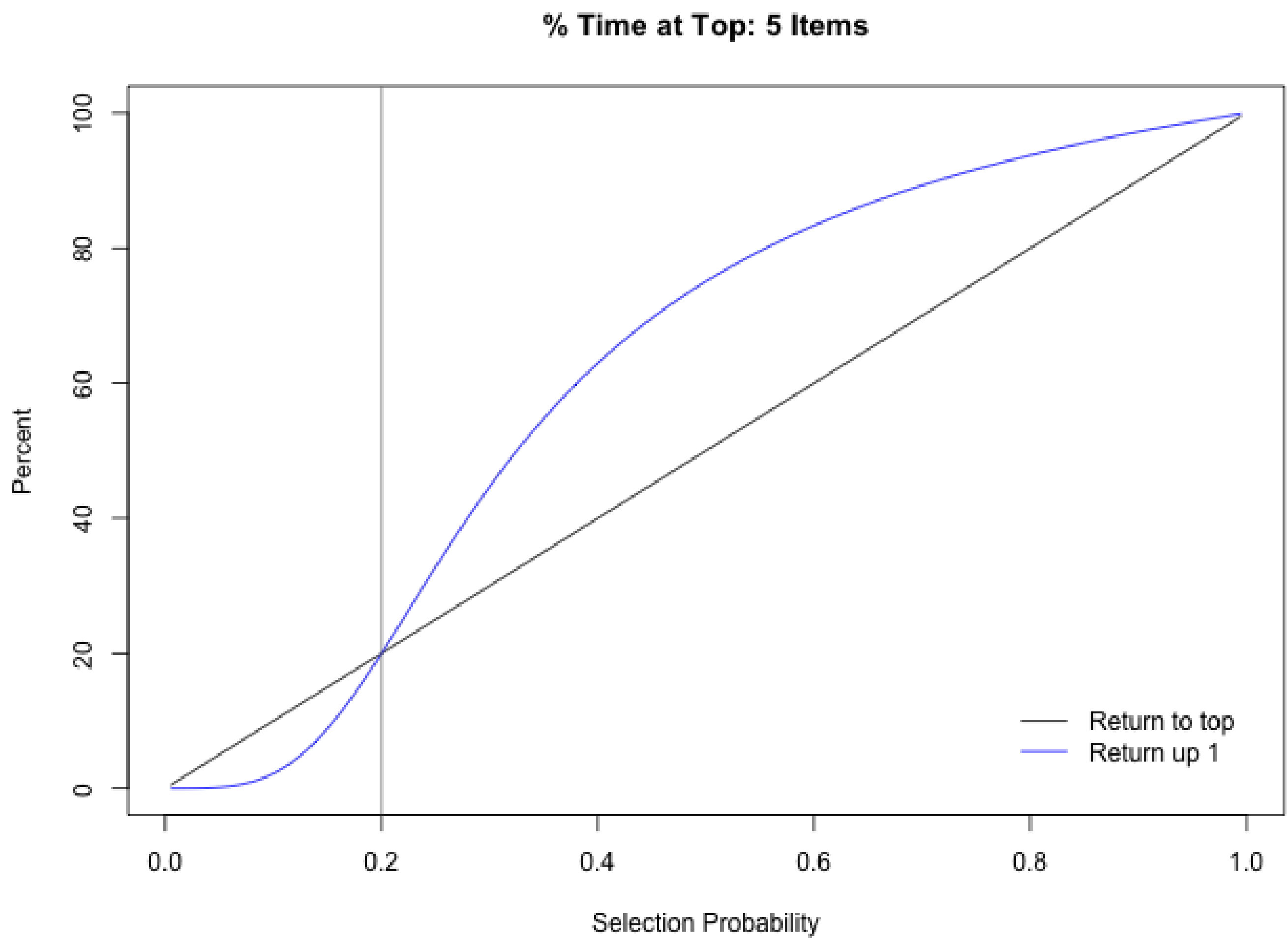
If $p \neq q$ then

$$\pi_1 = \frac{1}{\sum_{k=0}^{4} \left(\frac{q}{p}\right)^k} = \frac{1 - \frac{p}{q}}{1 - \left(\frac{q}{p}\right)^5}.$$

and

$$\pi_k = \left(\frac{q}{p}\right)^{k-1} \frac{1 - \frac{p}{q}}{1 - \left(\frac{q}{p}\right)^5}, \ 1 \leq k \leq 5.$$

Here is a graphical comparison of these two algorithms. It shows that the second algorithm performs better than the first. When $p < 0.2$, drive A is less likely to be selected than the other drives. In this case Algorithm 2 results in a lower proportion of time spent by A at the top. However, when $p > 0.2$, drive A is more likely to be selected and Algorithm 2 results in a higher proportion of time spent by A at the top.



% Time at Top: 5 Items
Percent
Selection Probability
0
20
40
60
80
100
0.0
0.2
0.4
0.6
0.8
1.0
Return to top
Return up 1

Bernoulli and Poisson Processes

An important family of Markov chains is generated by sums of i.i.d. random variables. Let X_i, $i \geq 1$, be i.i.d. discrete random variables assumed WLOG to be integer valued with pmf,

$$P(X_i = k) = p_k, \quad \sum_k p_k = 1.$$

Define

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \; n \geq 1.$$

Then it is easily seen that $\{S_n\}$ is a homogeneous Markov chain with transition probabilities

$$\begin{aligned} P[i,j] &= P(S_{n+1} = j | S_n = i) \\ &= P(S_n + X_{n+1} = j | S_n = i) \\ &= P(X_{n+1} = j - i | S_n = i) \\ &= p_{j-i}. \end{aligned}$$

Note that the last equality holds since X_{n+1} and X_k, $1 \leq k \leq n$ are independent. In most cases we are not much interested in the MC behavior of such processes, but there are other properties we will examine. These properties will be discussed for some special examples and extensions.

Bernoulli Processes

Let $\{X_n\}$, $n \geq 1$, denote a sequence of independent Bernoulli random variables with the same success probability p. This sequence can be used to represent a two-state system in which a state occurs randomly and independently in discrete time. For example, the occurrence or non-occurrence of a particular type of defect in a wafer could be modelled in this way. Departures from this model might indicate the presence of a systematic error in the production process.

There are two other ways to represent the Bernoulli Process. Define $S_0 = 0$ and for $n \geq 1$, let

$$S_n = \sum_{k=1}^{n} X_k.$$

The collection, $\{S_n, \; n \geq 0\}$, defines a discrete time, discrete state space stochastic process. It is the counting process representation of the Bernoulli process since S_n represents the number of successes up to time n. Note that these random variables are not independent. The other way to represent this process is to record the numbers of failures between successes. Let D_1 denote the number of failures before the first success and for $n > 1$, let D_n denote the number of failures after the $(n-1)^{th}$ success but before the n^{th} success. This sequence of random variables is referred to as the inter-arrival time representation of the Bernoulli process. If one of these three sequences of random variables is known, then the other two can be derived from the known sequence. Note that if T_N is the trial on which the n^{th} success occurs then,

$$T_n = n + \sum_{k=1}^{n} D_k.$$

Properties of S_n and D_n

1. The distribution of S_n is $Binomial(n,p)$. This follows directly from the definition of S_n.

2. If $m > n \geq 0$ then $S_m - S_n$ and S_n are independent random variables. Note that

$$S_n = \sum_{k=1}^{n} X_k, \quad S_m - S_n = \sum_{k=n+1}^{m} X_k.$$

   Since S_n and $S_m - S_n$ are functions of non-overlapping sets of X_k's, then S_n and $S_m - S_n$ are independent. This argument can be extended to prove that if $r \geq 1$ and $0 \leq n_0 < n_1 < \cdots < n_r < \infty$, then $S_{n_0}, S_{n_1} - S_{n_0}, \cdots, S_{n_r} - S_{n_{r-1}}$ are independent random variables. If this property is satisfied for every $r \geq 1$, then we say that the stocastic process has **independent increments**.

3. Let $m > n$. Then the distribution of $S_m - S_n$ is $Binomial(m-n,p)$. This follows from the fact that $S_m - S_n$ is the sum of $m-n$ independent Bernoulli random variables all with the same success probability p. In particular note that this distribution does not depend directly on the time points m or n, but instead only depends on the length of the time interval, $m-n$. Any stochastic process that possesses this property for all $m > n$ is said to have **stationary increments**.

4. Let $m > n$. Then $E(S_m|S_n) = S_n + (m-n)p$.
   Proof:

$$\begin{aligned} E(S_m|S_n = k) &= E(S_m - S_n + S_n|S_n = k) \\ &= E(S_m - S_n|S_n = k) + E(S_n|S_n = k) \\ &= E(S_m - S_n) + k = (m-n)p + k. \end{aligned}$$

   This follows since $S_m - S_n$ and S_n are independent. Now substitute S_n for k to obtain the desired result.

5. Let $m > n$. Then $S_M - S_n$ and S_n are independent since they involve non-overlapping time intervals, and so,

$$\begin{aligned} E(S_m S_n) &= E((S_m - S_n + S_n)S_n) \\ &= E((S_m - S_n)S_n) + E(S_n^2) \\ &= E(S_m - S_n)E(S_n) + Var(S_n) + (E(S_n))^2 \\ &= (m-n)pnp + np(1-p) + n^2p^2 \\ &= n(m-n)p^2 + np(1-p) + n^2p^2 \\ &= nmp^2 + np(1-p). \end{aligned}$$

This implies that

$$Cov(S_m, S_n) = E(S_m S_n) - E(S_m)E(S_n) = np(1-p)$$

which does not depend on m. Also,

$$Cor(S_m, S_n) = \frac{Cov(S_m, S_n)}{\sqrt{mpq}\sqrt{npq}} = \sqrt{\frac{n}{m}}$$

which does not depend on p.

6. The distribution of D_1 is *Geometric*(p). This follows directly from the definition of D_1. Note that the version of the geometric distribution for this property is the one that counts the number of failures before the first success.

7. $P(D_1 = k, D_2 = r) = pq^k pq^r,\ k \geq 0,\ r \geq 0.$
Proof:

$$\begin{aligned} P(D_1 = k, D_2 = r) &= P(X_1 = 0, \cdots, X_k = 0, X_{k+1} = 1, X_{k+2} = 0, \cdots, \\ &\qquad X_{k+r+1} = 0, X_{k+r+2} = 1) \\ &= q^k pq^r p. \end{aligned}$$

This property implies that D_1, D_2 are independent geometric random variables. This argument can be extended to show that $\{D_n,\ n \geq 0\}$ is a sequence of independent geometric random variables with the same success probability p.

8.

$$\begin{aligned} P(D_1 = r | S_n = 1) &= \frac{1}{n},\ 0 \leq r < n, \\ P(T_1 = k | S_n = 1) &= P(D_1 = k-1 | S_n = 1) = \frac{1}{n},\ 1 \leq k \leq n, \end{aligned}$$

In English this property says that given there was 1 success up to time n, the time of that success is uniformly distributed over the time points, $1, \cdots, n$.
Proof:

$$\begin{aligned} P(D_1 = r | S_n = 1) &= \frac{P(X_1 = 0, \cdots, X_r = 0, X_{r+1} = 1, X_{r+2} = 0, \cdots, X_n = 0)}{P(S_n = 1)} \\ &= \frac{q^r p q^{n-r-1}}{np^1 q^{n-1}} \\ &= \frac{1}{n}. \end{aligned}$$

This also shows that $E(D_1 | S_n = 1) = (n-1)/2$.

9. **Renewal Property**. Suppose that a success has occurred at time r. We can define a new stochastic process by effectively resetting the number of successes to 0 and resetting the clock to 0. The resulting stochastic process would be $S_n^* = S_{n+r} - S_r$. Since

$$S_{n+r} - S_r = \sum_{k=1}^{n} X_{r+k},$$

then this new process is defined in the same way as the original process, and so $\{S_n^*\}$ also is a Bernoulli process with the same properties as the original process. Therefore, if we are interested in observing an ongoing Bernoulli process and we synchronize the start of our observations at the time of a success, then the process we observe will be equivalent to the original Bernoulli process.

10. **Renewal Property, continued**. Now suppose that we begin our observation of the Bernoulli process at some arbitrary time r. We know from the previous result that starting from the time of the next success, we will be observing a Bernoulli process with the same properties as the original. Therefore, the only possible difference between the properties of the process we observe and the properties of the original process would be the time until the first success in the process we observe. Let D_1^* denote the number of failures until the first success after time r. Then

$$\begin{aligned} P(D_1^* = k) &= P(X_{r+1} = 0, \cdots, X_{r+k} = 0, X_{r+k+1} = 1) \\ &= pq^k. \end{aligned}$$

Hence, the distribution of D_1^* is the same geometric distribution as all the other inter-arrival times, and so the process we observe beginning at an arbitrary time r is a Bernoulli process with the same properties as the original process. Note that this also shows that if we begin observing an ongoing Bernoulli process at an arbitrary time, the inter-arrival time that contains the start of our observations is special. In particular, the time between the success prior to the start of our observations and the time of the next success has a different distribution than the other inter-arrival times since there may have been some failures between the previous success and the start of our observations.

11. **Markov Property**. Think of time $n+1$ as the future and time n as the present, and consider the conditional distribution of S_{n+1} given the present and the past,

$$P(S_{n+1} = k | S_1 = k_1, \cdots, S_n = k_n).$$

Since $S_{n+1} = X_{n+1} + S_n$ and since X_{n+1} and $\{S_1, \cdots, S_n\}$ are independent, then

$$\begin{aligned} P(S_{n+1} = k | S_1 = k_1, \cdots, S_n = k_n) &= P(X_{n+1} + S_n = k | S_1 = k_1, \cdots, S_n = k_n) \\ &= P(X_{n+1} + k_n = k | S_1 = k_1, \cdots, S_n = k_n) \\ &= P(X_{n+1} = k - k_n). \end{aligned}$$

Therefore,

$$P(S_{n+1} = k | S_1, \cdots, S_n) = P(S_{n+1} = k | S_n).$$

This is the Markov property for stochastic processes. In English it means that the conditional distribution of S_{n+1} given the present and the past depends only on the present. Discrete-time stochastic processes that satisfy this property are said to be **Markov processes**.

Poisson Processes

The next example of a stochastic process we will examine is the Poisson process. This stocastic process has many important applications in addition to providing the basis for extensions that make it even more widely applicable. The Poisson process is a continuous time, discrete state space process, $\{N(t),\ t \geq 0\}$, that represents the number of arrivals of some entity up to time t. It is defined by

1. $\{N(t),\ t \geq 0\}$ is a counting process; that is, $N(t)$ is non-negative integer-valued, $N(0) = 0$, and if $t > s$, then $N(t) \geq N(s)$ with probability 1.

2. $\{N(t),\ t \geq 0\}$ has stationary, independent increments. Specifically, for every $r \geq 1$ and for every collection $0 \leq t_0 < t_1 < \cdots < t_r$, $N(t_1) - N(t_0), \cdots, N(t_r) - N(t_{r-1})$ is a collection of independent random variables. For every $0 \leq s < t < \infty$, the distribution of $N(t) - N(s)$ depends only on $t - s$ and therefore is the same as the distribution of $N(t - s)$.

3. The likelihood that there is exactly one arrival during a small interval of time is proportional to the length of the time interval. That is, there exists $0 < \lambda < \infty$ such that

$$\lim_{h \searrow 0} \frac{P(N(t+h) - N(t) = 1)}{h} = \lambda.$$

The parameter λ is called the intensity of the Poisson process. The likelihood that there is more than 1 arrival during a small interval of time is vanishingly small. Specifically,

$$\lim_{h \searrow 0} \frac{P(N(t+h) - N(t) > 1)}{h} = 0.$$

It can be shown that condition (3) is equivalent to the condition that the distribution of $N(t)$ is Poisson with mean λt. This condition also shows that the Poisson process is not an appropriate model for congested arrivals such as traffic on LBJ Expressway.

$N(t)$ is the counting process representation of the Poisson process. There is also an arrival time representation. Let T_n denote the time of the n^{th} arrival and let $D_1 = T_1$, $D_n = T_n - T_{n-1}$, $n > 1$ denote the inter-arrival times. The correspondence between the counting process representation and the arrival times is given by,

$$\{N(t) \geq n\} = \{T_n \leq t\}.$$

In particular,

$$P(D_1 \leq t) = P(T_1 \leq t) = P(N(t) \geq 1) = 1 - e^{-\lambda t}.$$

This shows that the time of the first arrival has an exponential distribution with mean $\mu = 1/\lambda$.

Properties of $N(t)$ and T_n.

1. Let $0 < s < t$. Then

$$\begin{aligned} P(T_1 > s, T_2 > t) &= P(N(s) < 1, N(t) < 2) \\ &= P(N(s) = 0, N(t) - N(s) \leq 1) \\ &= P(N(s) = 0)P(N(t) - N(s) \leq 1) \\ &= e^{-\lambda t}[1 + \lambda(t - s)]. \end{aligned}$$

Differentiating with respect to s, t gives the joint density of T_1, T_2,

$$f(s,t) = \lambda^2 e^{-\lambda t},\ 0 < s < t < \infty.$$

The joint moment generating function of T_1, T_2 can be obtained from this joint density and is equal to

$$M(\theta, \eta) = E[e^{\theta T_1 + \eta T_2}] = \frac{\lambda^2}{(\lambda - \theta - \eta)(\lambda - \eta)}.$$

Finally, the joint moment generating function of D_1, D_2 is then

$$\begin{aligned} E[e^{\theta_1 D_1 + \theta_2 D_2}] &= E[e^{\theta_1 T_1 + \theta_2 (T_2 - T1)}] \\ &= M(\theta_1 - \theta_2, \theta_2) \\ &= \frac{\lambda^2}{(\lambda - \theta_1)(\lambda - \theta_2)} \\ &= \left(\frac{\lambda}{\lambda - \theta_1}\right)\left(\frac{\lambda}{\lambda - \theta_2}\right). \end{aligned}$$

This shows that D_1, D_2 are independent *exponential*(λ) random variables. This argument can be extended to show that the inter-arrival times, $\{D_n,\ n \geq 1\}$ are independent *exponential*(λ) random variables.

2. The distribution of T_n, the time of the n^{th} arrival, is $Gamma(n, \lambda)$. This follows from the previous property since

$$T_n = \sum_{k=1}^{n} D_k$$

and the sum of independent exponential random variables with the same parameter has a *gamma* distribution. This result gives an alternative definition of a Poisson process: a Poisson process is a counting process on $[0, \infty)$ with inter-arrival times that are independent, exponentially distributed with parameter λ.

3. Let $0 < s < t$. Then

$$P(T_1 \leq s | N(t) = 1) = \frac{s}{t}.$$

This says that conditioned on the event that there is exactly 1 arrival up to time t, the time of that arrival is uniformly distributed over the interval $[0, t]$.

$$\begin{aligned} P(T_1 \leq s | N(t) = 1) &= \frac{P(T_1 \leq s, N(t) = 1)}{P(N(t) = 1)} \\ &= \frac{P(N(s) = 1, N(t) = 1)}{P(N(t) = 1)} \\ &= \frac{P(N(s) = 1, N(t) - N(s) = 0)}{P(N(t) = 1)} \\ &= \frac{P(N(s) = 1)P(N(t) - N(s) = 0)}{P(N(t) = 1)} \\ &= \frac{\lambda s e^{-\lambda t}}{\lambda t e^{-\lambda t}} \\ &= \frac{s}{t}. \end{aligned}$$

This argument can be extended to show that conditioned on the event that $N(t) = n$, the n arrival times have a distribution that is the distribution of the ordered values from a sample of size n uniformly distributed random variables over the interval $[0, t]$.

4. For $0 < s < t$,

$$P(N(s) = k|N(t) = n) = \binom{n}{k} p^k q^{n-k},$$

where $p = s/t$ and $q = 1 - p$. That is, if $0 < s < t$, then the conditional distribution of $N(s)$ given $N(t)$ is $Binomial(n, p)$.

5. **Renewal Property**. Suppose that we begin observing a Poisson process at the time of an arrival, say T_k. Let N_k denote this new process, $N_k(t) = N(T_k + t) - k$. Since the inter-arrival times of this new process are independent, exponentially distributed with the same parameter λ, then N_k is a Poisson process with the same properties as the original process.

6. **Renewal Property, continued**. Now suppose that we begin observing the Poisson process at an arbitrary time, t_0, and define a new process by $N_0(t) = N(t_0+t)-N(t_0)$. The inter-arrival times after the first arrival of N_0 are independent, exponentially distributed, so the only possible difference between this process and the original would be the distribution of the time to the first arrival of the new process. This corresponds to $T_{01} = T_{N(t_0)+1} - t_0$, called the residual waiting time. It can be shown using the memory-less property of the exponential distribution that this time also has the same exponential distribution as the other inter-arrival times and is independent of those times, so that this new process is also a Poisson process with the same properties as the original. Note that as was the case in the Bernoulli process, this implies that the inter-arrival time that contains our starting time, t_0, is special in that it is stochastically longer than the other inter-arrival times.

7. **Markov Property**. Consider time points $0 \leq t_0 < t_1 < \cdots < t_{n+1}$ and the conditional distribution of the future given the present and the past,

$$P(N(t_{n+1}) = k|N(t_1) = k_1, \cdots, N(t_n) = k_n).$$

Since $N(t_{n+1}) = [N(t_{n+1}) - N(t_n)] + N(t_n)$, and since $N(t_{n+1}) - N(t_n)$ is independent of $\{N(t_1), \cdots, N(t_n)\}$ (independent increments), then

$$\begin{aligned}
P(N(t_{n+1}) = k|N(t_1) = k_1, \cdots, N(t_n) = k_n) = \\
P([N(t_{n+1}) - N(t_n)] + N(t_n) = k|N(t_1) = k_1, \cdots, N(t_n) = k_n) \\
= P([N(t_{n+1}) - N(t_n)] + k_n = k|N(t_1) = k_1, \cdots, N(t_n) = k_n) \\
= P(N(t_{n+1}) - N(t_n) = k - k_n).
\end{aligned}$$

Hence,

$$\begin{aligned} P(N(t_{n+1}) = k | N(t_1), \cdots, N(t_n)) &= P(N(t_{n+1}) - N(t_n) = k - N(t_n)) \\ &= P(N(t_{n+1}) = k | N(t_n)). \end{aligned}$$

That is, the conditional distribution of the future given the present and the past depends only on the present. This is the Markov property for continuous-time processes.

8. If $N_1, \cdots, N_r$ are independent Poisson processes with intensities $\lambda_1, \cdots, \lambda_r$, respectively, then $N = \sum_1^r N_k$ is a Poisson process with intensity $\lambda = \sum_1^r \lambda_k$. The proof of this property follows from the result derived earlier that a sum of independent Poisson random variables has a Poisson distribution. This process of summing a collection of stocastic processes to form a new process is referred to as the **superposition** of the processes.

9. Suppose that the Poisson process N represents arrivals of customers to a service center that contains r queues and that there is a monitor that decides which queue a new arrival will enter. Suppose also that this decision is made independently of the arrival process and independently of the queue assignments of other arrivals such that the probability that an arrival is assigned to queue k is p_k, where $\sum p_k = 1$. Let $N_k(t)$ denote the number of arrivals to queue k up to time t, $1 \leq i \leq r$. Then $N_1, \cdots, N_r$ are independent Poisson processes with intensities $\lambda_k = p_k \lambda$. This process of splitting the arrivals of a Poisson process is referred to as **thinning**.

   We can represent thinning as follows. Let $X_i,\ i \geq 1$ be independent random variables that are also independent of the arrival process N with

   $$P(X_i = 1) = p_k,\ 1 \leq k \leq r.$$

   Then $N_k(t) = 0$ if $N(t) = 0$; otherwise

   $$N_k(t) = \sum_{i=1}^{N(t)} I\{X_i = k\}.$$

   We can think of the arrivals as marked by the X_i's, and these marks indicate into which queue the arrival is sent.

   This representation of thinning give a generalization of Poisson processes called compound Poisson processess. Suppose $N(t),\ t \geq 0$, is a Poisson process with intensity λ and $X_i,\ i \geq 1$ are i.i.d. non-negative random variables that are also independent of the arrival process N. Define $X_0 = 0$, and let

   $$X(t) = \sum_{k=0}^{N(t)} X_k,\ t \geq 0.$$

Then $X(t),\ t \geq 0$ is a compound Poisson process. We can think of this as a Poisson arrival process, $N(t)$, in which the k^{th} arrival has associated with it a value X_k, and $X(t)$ represents the total value of all arrivals up to time t. Note that this is a random sum of random variables, so

$$\begin{aligned} E[X(t)] &= E[N(t)]E(X_1) = \mu\lambda t, \\ Var[X(t)] &= \sigma^2\lambda t + \mu^2\lambda t = (\sigma^2 + \mu^2)\lambda t, \end{aligned}$$

where $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$. More generally, suppose that M_X is the MGF of X_1. Then

$$\begin{aligned} E[\exp\{\theta X(t)\}|N(t) = n] &= E[\exp\{\theta \sum_{k=0}^{n} X_k\}] \\ &= E[\prod_{k=0}^{n} \exp\{\theta X_k\}] \\ &= \prod_{k=0}^{n} E[\exp\{\theta X_k\}] \\ &= [M_X(\theta)]^n. \end{aligned}$$

This implies that

$$\begin{aligned} E[\exp\{\theta X(t)\}|N(t)] &= [M_X(\theta)]^{N(t)} \\ &= \exp\{N(t)\log(M_X(\theta))\}. \end{aligned}$$

And so,

$$\begin{aligned} E[\exp\{\theta X(t)\}] &= E(E[\exp\{\theta X(t)\}|N(t)]) \\ &= E[\exp\{N(t)\log(M_X(\theta))\}] \\ &= \exp\{\lambda t[M_X(\theta) - 1]\}. \end{aligned}$$

We can perform other thinning-related operations. For example, let A denote a subset of $[0, \infty)$, and define

$$\begin{aligned} N_A(t) &= \sum_{k=0}^{N(t)} I(X_k \in A) \\ X_A(t) &= \sum_{k=0}^{N(t)} X_k I(X_k \in A). \end{aligned}$$

Then $N_A(t)$ represents the number of arrivals up to time t whose values are in A, and $X_A(t)$ represents the total value of those arrivals.

Example 5.21, p. 334. Suppose insurance claims arrive according to a Poisson process with intensity λ and the claim amounts associated with those claims are i.i.d. with mean μ and are independent of the claim arrival times (reasonable under non-disaster conditions). Also suppose the value covered by the policy is depreciated over time. That is, the claim amount C is reduced to

$$Ce^{-\alpha T}$$

for a claim made at time T. Then the total cost of all claims up to time t is

$$D(t) = \sum_{i=1}^{N(t)} C_i e^{-\alpha S_i},$$

where S_i is the arrival time of the *i-th* claim, C_i is the amount of that claim, and *N(t)* is the number of claims up to time t. Since this is a random sum of random variables, we can obtain its expected value by conditioning on *N(t)*.

$$E[D(t)] = \sum_{n=0}^{\infty} E[D(t)|N(t) = n]P(N(t) = n) = \sum_{n=0}^{\infty} E[D(t)|N(t) = n]\frac{(\lambda t)^n}{n!}e^{-\lambda t}.$$

Recall that conditioned on $N(t) = n$, the times of the n arrivals are distributed as the ordered values of n i.i.d. uniform r.v.'s on $(0, t)$. Let $U_1, \cdots, U_n$ denote the i.i.d. uniform r.v.'s and let $U_{(1)}, \cdots, U_{(n)}$ denote their ordered values. Then

$$\begin{aligned}
E[D(t)|N(t) = n] &= E\left[\sum_{i=1}^{n} C_i e^{-\alpha U_{(i)}}\right] \\
&= \sum_{i=1}^{n} E\left[C_i e^{-\alpha U_{(i)}}\right] \\
&= \sum_{i=1}^{n} E[C_i]E\left[e^{-\alpha U_{(i)}}\right] \quad \text{(independence of claim amounts and times)} \\
&= \mu \sum_{i=1}^{n} E\left[e^{-\alpha U_{(i)}}\right] \\
&= \mu E\left[\sum_{i=1}^{n} e^{-\alpha U_{(i)}}\right] \\
&= \mu E\left[\sum_{i=1}^{n} e^{-\alpha U_i}\right] = n\mu E\left[e^{-\alpha U}\right],
\end{aligned}$$

where the last step follows from the fact that a sum of ordered values is the same as the sum of the unordered values. Since U is uniformly distributed, then

$$E\left[e^{-\alpha U}\right] = \frac{1}{t}\int_0^t e^{-\alpha s} ds = \frac{1}{\alpha t}(1 - e^{-\alpha t}).$$

Therefore,

$$E[D(t)|N(t)] = N(t)\frac{\mu}{\alpha t}(1 - e^{-\alpha t}),$$

and so

$$E[D(t)] = \lambda t\frac{\mu}{\alpha t}(1 - e^{-\alpha t}) = \lambda\frac{\mu}{\alpha}(1 - e^{-\alpha t}).$$

Markov Shot Noise Process. A similar problem is a model for noise which arrives at discrete times but whose value decays over time. Specifically, assume noise events arrive according to a Poisson process $N(t)$ with intensity λ. The initial magnitude of the i^{th} noise event is a r.v. X_i and the noise magnitude at time t is

$$X_i e^{-\alpha(t-S_i)},$$

where S_i is the arrival time of the i^{th} noise event and α is a constant. We assume $\{X_i,\ i \geq 1\}$ are i.i.d., are independent of $N(t)$, $t > 0$, and are independent of the arrival process. Then the total noise at time t is given by

$$X(t) = \sum_{i=1}^{N(t)} X_i e^{-\alpha(t-S_i)},$$

for $N(t) > 0$, and $X(t) = 0$ when $N(t) = 0$. This model is referred to as a *shot noise* process.

The moment generating function of this process can be obtained by conditioning on $N(t)$ and using the property that the conditional distribution of the arrival times given $N(t) = n$ is the distribution of the ordered values of n i.i.d. uniform r.v.'s. Then

$$E[\exp\{\theta X(t)\}|N(t) = n] = E\left[\exp\left(\theta\sum_{i=1}^{n} X_i e^{-\alpha(t-U_{(i)})}\right)\right],$$

where $U_1, \ldots U_n$ are i.i.d uniform r.v.s on $(0, t)$, and $U_{(1)}, \ldots U_{(n)}$ are their ordered values. Since X_i are i.i.d., U_i are i.i.d. and are independent of X_i, then

$$\sum_{i=1}^{n} X_i e^{-\alpha(t-U_{(i)})}$$

and

$$\sum_{i=1}^{n} X_i e^{-\alpha(t-U_i)}$$

have the same distribution. Therefore,

$$E[\exp\{\theta X(t)\}|N(t) = n] = \left(E\left[X_i e^{-\alpha(t-U_i)}\right]\right)^n.$$

Let $M_X(\theta)$ denote the m.g.f. of X_i and note that the expectation in the right-hand-side of the previous equation refers to the joint distribution of X_i, U_i which are independent. Hence,

$$\begin{aligned} E\exp\{\theta X e^{-\alpha(t-U)}\} &= E\left[\frac{1}{t}\int_0^t \exp\{\theta X_i e^{-\alpha(t-u)}\}du\right] \\ &= \frac{1}{t}\int_0^t E\left[\exp\{\theta X e^{-\alpha y}\}\right]dy \quad (y = t-u) \\ &= \frac{1}{t}\int_0^t M_X\left(\theta e^{-\alpha y}\right) \\ &= B. \end{aligned}$$

This gives

$$\begin{aligned} E\exp\{\theta X(t)\} &= \sum_{n=0}^{\infty} B^n \frac{(\lambda t)^n}{n!} \\ &= e^{\lambda t(B-1)} \\ &= \exp\left(\lambda \int_0^t [M_X(\theta e^{-\alpha y}) - 1]dy\right) \end{aligned}$$

Moments can be obtained by differentiating the MGF. So for example, if $E(X) = \mu$ and $Var(X) = \sigma^2$, then

$$E[X(t)] = \frac{\lambda\mu}{\alpha}(1 - e^{-\alpha t}),$$

$$Var[X(t)] = \frac{\lambda(\mu^2 + \sigma^2)}{2\alpha}(1 - e^{-2\alpha t}),$$

$$\begin{aligned} Cov(X(t), X(t+s)) &= e^{-\alpha s} Var[X(t)] \\ &= e^{-\alpha s}\frac{\lambda(\mu^2 + \sigma^2)}{2\alpha}(1 - e^{-2\alpha t}). \end{aligned}$$

Also, we can obtain the limiting distribution of $X(t)$ by letting $t \to \infty$ in the MGF. This gives

$$\lim_{t\to\infty} E[\exp\{\theta X(t)\}] = \exp\left\{\lambda \int_0^{\infty} [M_X(\theta e^{-\alpha y}) - 1]dy\right\}.$$

Now consider the special case in which the X_i are exponentially distributed with rate η. Then

$$M_X(\theta) = \frac{\eta}{\eta - \theta}$$

and so

$$\begin{aligned}
\lim_{t\to\infty} E[\exp\{\theta X(t)\}] &= \exp\left\{\lambda \int_0^\infty \left(\frac{\eta}{\eta-\theta e^{-\alpha y}} - 1\right) dy\right\} \\
&= \exp\left\{\frac{\lambda}{\alpha}\int_0^\theta \frac{1}{\eta - x} dx\right\} \quad (x = \theta e^{-\alpha y}) \\
&= \left(\frac{\eta}{\eta-\theta}\right)^{\lambda/\alpha}.
\end{aligned}$$

This is the MGF of a gamma r.v. with shape λ/α and rate η.

Renewal Processes

The renewal property of Poisson processes provides a third definition of a Poisson process: a counting process with interarrival times that are independent and have the same exponential distribution with rate λ. We can generalize this definition to define other counting processes whose interarrival times are i.i.d r.v.'s, $X_k,\ k \geq 1$. The distribution of the interarrival times is defined over $[0, \infty)$ with

$$P(X_k > 0) > 0.$$

Such processes are called *renewal processes*. For example, every other arrival in a Poisson process is a renewal process, but not a Poisson process, because the interarrival times for those arrivals have a *Gamma* distribution with shape *2*.

Obviously renewal processes satisfy the renewal property when the process is reset at the time of an arrival. However, the Poisson process is the only renewal process in which time to the first arrival after resetting at an arbitrary time t has the same exponential distribution as all the other interarrival times. To allow for resetting at arbitrary times, the first arrival time, X_1, is allowed to have a different distribution, but $X_k,\ k \geq 2$, are i.i.d. We will only consider renewal processes in which all interarrival times have the same distribution. Let F denote the d.f. of the interarrival times. Let

$$\mu = E(X_n),\ n \geq 1.$$

Since

$$F(0) = P(X_k = 0) < 1,$$

then $\mu > 0$. We will assume that $\mu < \infty$.

The arrival or renewal times are defined by

$$S_0 = 0, \quad S_n = \sum_{k=1}^n X_k,\ n \geq 1.$$

The counting process version is given by

$$N(t) = \max\{n :\ S_n \leq t\}.$$

The Strong Law of Large Numbers implies that

$$\frac{S_n}{n} \to \mu \text{ as } n \to \infty$$

with probability 1. This also implies that $N(t) < \infty$ for $0 < t < \infty$ and that

$$N(\infty) = \lim_{t\to\infty} N(t) = \infty$$

with probability 1.

Let X, Y be independent r.v.'s with distribution functions F_X, F_Y, respectively, and assume they have density functions f_X, f_Y. The convolution of F_X, F_Y is defined to be the distribution of the sum of the r.v.s and is denoted by $F_X * F_Y$,

$$\begin{aligned} F_X * F_Y(t) &= P(X + Y \leq t) \\ &= \int_0^\infty P(X \leq t - s | Y = s) f_Y(s) ds \\ &= \int_0^\infty F_X(t-s) f_Y(s) ds \\ &= \int_0^\infty \left[\int_0^{t-s} f_X(u) du \right] f_Y(s) ds. \end{aligned}$$

The density function of the convolution is

$$f_{X+Y}(t) = \int_0^\infty f_X(t-s) f_Y(s) ds.$$

If F is the d.f. of the interarrival times of a renewal process, then the distribution of

$$S_n = \sum_{k=1}^n X_k$$

is

$$F_n(t) = F^{*n}(t) = F * \cdots * F(t).$$

Since

$$\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\},$$

then

$$P(N(t) = n) = P(S_n \leq t) - P(S_{n+1} \leq t) = F_n(t) - F_{n+1}(t).$$

An alternative expression for the p.m.f. of *N(t)* can be obtained by conditioning on S_n.

$$\begin{aligned} P(N(t)=n) &= \int_0^\infty P(N(t)=n|S_n=y)f_n(y)dy \\ &= \int_0^t P(X_{n+1}>t-y|S_n=y)f_n(y)dy \\ &= \int_0^t (1-F(t-y))f_n(y)dy. \end{aligned}$$

In the case of a Poisson process, F is the exponential distribution with rate λ and F_n is the gamma distribution with shape n and rate λ. Substitution of these distributions into the above integral gives the Poisson p.m.f. for *N(t)*.

One of the major results for renewal processes concerns the mean function,

$$\begin{aligned} m(t) &= E(N(t)) \\ &= \sum_{n=1}^\infty P(N(t)\geq n) \\ &= \sum_{n=1}^\infty P(S_n\leq t) \\ &= \sum_{n=1}^\infty F_n(t) \\ &= \sum_{n=1}^\infty F^{*n}(t). \end{aligned}$$

This function is referred to as the renewal function and it uniquely determines the interarrival time distribution and, hence, the renewal process.

The renewal function satisfies an integral equation,

$$m(t) = F(t) + \int_0^t m(t-s)f(s)ds,$$

called the renewal equation. This equation is obtained by conditioning on the time of the first renewal. The renewal property implies that for $0<s<t$,

$$E[N(t)|X_1=s] = 1 + E[N(t-s)]$$

and

$$E[N(t)|X_1=s] = 0,\ \ s>t.$$

This gives

$$m(t) = E[N(t)] \ = \ \int_0^\infty E[N(t)|X_1=s]f(s)ds$$

$$
\begin{aligned}
&= \int_0^t [1 + m(t-s)] f(s) ds \\
&= F(t) + \int_0^\infty m(t-s) f(s) ds.
\end{aligned}
$$

This equation can be solved in general for only a few cases. Of main interest is the limiting behavior of the renewal function.

First note that

$$S_{N(t)} \leq t < S_{N(t)+1},$$

and so

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}.$$

The Strong Law of Large Numbers implies that

$$\frac{S_{N(t)}}{N(t)} \to \mu$$

and

$$\frac{S_{N(t)+1}}{N(t)} \to \mu$$

with probability 1. Therefore,

$$\frac{N(t)}{t} \to \frac{1}{\mu}$$

with probability 1 as $t \to \infty$. The *Elementary Renewal Theorem* states that this limit also holds for the mean function,

$$\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

The first step in proving this theorem is to obtain an integral equation for the mean time to first renewal after time t. This equation is similar to the renewal equation. Note that the time to first renewal after time t is

$$S_{N(t)+1},$$

and so its expected value is given by

$$
\begin{aligned}
g(t) &= E[S_{N(t)+1}] \\
&= \int_0^\infty E[S_{N(t)+1} | X_1 = s] f(s) ds.
\end{aligned}
$$

It can be seen from the renewal property that

$$E[S_{N(t)+1}|X_1 = s] = \begin{cases} g(t-s)+s, & s < t \\ s, & s > t \end{cases}$$

This gives,

$$\begin{aligned} g(t) &= \int_0^\infty E[S_{N(t)+1}|X_1 = s]f(s)ds \\ &= \int_0^t [g(t-s)+s]f(s)ds + \int_t^\infty sf(s)ds \\ &= \int_0^t g(t-s)f(s)ds + \int_0^\infty sf(s)ds \\ &= \mu + \int_0^t g(t-s)f(s)ds. \end{aligned}$$

Now let

$$g_1(t) = \frac{g(t)}{\mu} - 1.$$

Then

$$\begin{aligned} g_1(t) &= \frac{\mu}{\mu} + \frac{1}{\mu}\int_0^t g(t-s)f(s)ds - 1 \\ &= \int_0^t \frac{g(t-s)}{\mu}f(s)ds \\ &= \int_0^t [g_1(t-s)+1]f(s)ds \\ &= F(t) + \int_0^t g_1(t-s)f(s)ds. \end{aligned}$$

This shows that $g_1(t)$ satisfies the renewal equation and so by uniqueness must equal the renewal function,

$$m(t) = g_1(t) = \frac{g(t)}{\mu} - 1.$$

This proves **Proposition 7.2**:

$$E[S_{N(t)+1}] = \mu[m(t)+1].$$

This proposition shows that the renewal function can be expressed in terms of the mean time to the next renewal,

$$m(t) = \frac{E[S_{N(t)+1}]}{\mu} - 1.$$

Let $Y(t)$ denote the waiting time to the next renewal after time t,

$$Y(t) = S_{N(t)+1} - t.$$

Then

$$t + E[Y(t)] = \mu[m(t) + 1],$$

and so

$$\frac{m(t)}{t} = \frac{1}{\mu} - \frac{1}{t} + \frac{E[Y(t)]}{t\mu}.$$

It can be shown that

$$\lim_{t\to\infty} \frac{E[Y(t)]}{t} = 0.$$

Therefore,

$$\begin{aligned}\lim_{t\to\infty} \frac{m(t)}{t} &= \lim_{t\to\infty}[\frac{1}{\mu} - \frac{1}{t} + \frac{E[Y(t)]}{t\mu}] \\ &= \frac{1}{\mu}.\end{aligned}$$

Example. Suppose the interarrival time distribution is heavy-tailed, for example,

$$f(t) = C(1+t)^{-\alpha},\ t > 0,$$

where $\alpha > 2$ and $C = \alpha - 1$. Then

$$\mu = E[X_1] = C\int_0^\infty t(1+t)^{-\alpha}dt = \frac{1}{\alpha - 2}.$$

Note that if $1 < \alpha \leq 2$, then $\mu = \infty$ in which case

$$\lim_{t\to\infty} \frac{m(t)}{t} = 0.$$

Otherwise, for large t,

$$m(t) \approx (\alpha - 2)t.$$

Example. Suppose a device receives shocks that arrive according to a Poisson process with intensity λ, the shocks cause damages that are i.i.d. r.v.'s with d.f. F, and the damages are independent of the arrival times. Also suppose the damage to the device is cumulative and the device fails when the total damage reaches or exceeds some threshold w. Let D_k denote the damage caused by the k-th shock, and let T_w denote the failure time of the device. Then

$$D(t) = \sum_{k=0}^{N(t)} D_k,$$

is the cumulative damage up to time t, where $D_0 = 0$, and

$$T_w = \min\{t: \ D(t) > w\}.$$

Then

$$\begin{aligned}
P(T_w > t) &= P(D(t) \le w) \\
&= E[P(D(t) \le w | N(t))] \\
&= \sum_{n=0}^{\infty} P(D(t) \le w | N(t) = n) \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \sum_{n=0}^{\infty} P(\sum_{k=0}^{n} D_k \le w) \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \sum_{n=0}^{\infty} F^{*n}(w) \frac{(\lambda t)^n}{n!} e^{-\lambda t}.
\end{aligned}$$

Also,

$$\begin{aligned}
E[T_w] &= \int_0^{\infty} P(T > t) dt \\
&= \sum_{n=0}^{\infty} F^{*n}(w) \int_0^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} dt \\
&= \sum_{n=0}^{\infty} F^{*n}(w) \frac{\lambda^n}{n!} \int_0^{\infty} t^n e^{-\lambda t} dt \\
&= \sum_{n=0}^{\infty} F^{*n}(w) \frac{\lambda^n}{n!} \frac{\Gamma(n+1)}{\lambda^{n+1}} \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} F^{*n}(w) \\
&= \frac{1}{\lambda} \left[1 + \sum_{n=1}^{\infty} F^{*n}(w) \right].
\end{aligned}$$

Note that

$$\sum_{n=1}^{\infty} F^{*n}(w)$$

is a renewal function.

Homework Assignments

Homework Assignment 1

Due date: Jan. 22, 2015.

Text, pages 15-20

1. Exercise 8
2. Exercise 10
3. Exercise 12 (note: assume experiment is repeated independently
4. Exercise 20
5. Exercise 21
6. Exercise 22
7. Exercise 29
8. Exercise 42

Solutions for Homework Assignment 1

1. Exercise 8.

$$1 \geq P(E\bigcup F) = P(E) + P(F) - P(E\bigcap F).$$

   This implies that

$$P(E\bigcap F) \geq P(E) + P(F) - 1.$$

2. Exercise 10. Proof by induction. Result holds trivially for *n=1*. Now assume

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i),$$

   and show result holds for *n+1*.

$$\begin{aligned} P\left(\bigcup_{i=1}^{n+1} E_i\right) &= P\left(\bigcup_{i=1}^{n} E_i \bigcup E_{n+1}\right) \\ &= \sum_{i=1}^{n} P(E_i) + P(E_{n+1}) - P\left(\bigcup_{i=1}^{n} E_i \bigcap E_{n+1}\right) \\ &= \sum_{i=1}^{n+1} P(E_i) - P\left(\bigcup_{i=1}^{n} E_i \bigcap E_{n+1}\right) \\ &\leq \sum_{i=1}^{n+1} P(E_i). \end{aligned}$$

3. Exercise 12 (note: assume experiment is repeated independently). Let

$$A = (E\bigcup F)^c$$

   and let

$$p = P(E\bigcup F) = P(E) + P(F).$$

   Then *P(A) = 1-p*. Sample space is
   $E, F, AE, AF, AAE, AAF, AAAE, AAAF, \ldots$, so

$$P(\overbrace{A\cdots A}^{n-1}E) = [P(A)]^{n-1}P(E),$$

and so

$$\begin{aligned}
P(\text{E } occurs\ before\ F) &= \sum_{n=1}^{\infty} P(\overbrace{A\cdots A}^{n-1}E) \\
&= \sum_{n=1}^{\infty}(1-p)^{n-1}P(E) \\
&= \frac{P(E)}{p} \\
&= \frac{P(E)}{P(E)+P(F)}.
\end{aligned}$$

4. Exercise 20

$$\begin{aligned}
P(same\ on\ exactly\ 2\ of\ 3\ dice) &= \sum_{k=1}^{6}[P(kkk^c)+P(kk^ck)+P(k^ckk)] \\
&= \sum_{k=1}^{6}[3(\frac{1}{6})(\frac{1}{6})(\frac{5}{6})] \\
&= \frac{15}{36}
\end{aligned}$$

5. Exercise 21

$$P(CB|M)=0.05,\quad P(CB|F)=0.0025,\quad P(M)=P(F)=0.5$$

So,

$$\begin{aligned}
P(M|CB) &= \frac{P(M\bigcap CB)}{P(CB)} \\
&= \frac{P(M\bigcap CB)}{P(CB|M)P(M)+P(CB|F)P(F)} \\
&= \frac{(0.05)(0.5)}{(0.05)(0.5)+(0.0025)(0.5)} \\
&= \frac{500}{525}=\frac{20}{21}
\end{aligned}$$

6. Exercise 22. Note that the game must end after *2n* games. This implies that after *2k* games, *k=1,2,...,n-1*, they must be tied, and that games *2n-1* and *2n* must be either AA or BB. Let

$$q=p^2+(1-p)^2=1-2p(1-p).$$

Then

$$\begin{aligned} P(\textit{game ends at } 2n) &= [2p(1-p)]^{n-1}[p^2+(1-p)^2] \\ &= q^{n-1}(1-q), \end{aligned}$$

and so

$$\begin{aligned} P(\text{A } \textit{wins}) &= \sum_{n=1}^{\infty} P(\textit{game ends at } 2n \textit{ and } A \textit{ wins}) \\ &= \sum_{n=1}^{\infty} [2p(1-p)]^{n-1}p^2 \\ &= \frac{p^2}{p^2+(1-p)^2}. \end{aligned}$$

7. Exercise 29
a) If E,F are mutually exclusive, then

$$P(E|F) = \frac{P(E\bigcap F)}{P(F)}$$

if $P(F) > 0$, otherwise $P(E|F)$ is undefined.
b) If $E \subset F$, then

$$P(E\bigcap F) = P(E) = 0.6,$$

and

$$P(F) \geq P(E) = 0.6.$$

Hence,

$$P(E|F) = \frac{P(E)}{P(F)} = \frac{0.6}{P(F)}.$$

c) If $F \subset E$, then $P(E\bigcap F) = P(F)$ and so $P(E|F) = 1$.

8. Exercise 42

$$P(H|2H\ coin) = 1, \quad P(H|\text{f}air\ coin) = 1/2, \quad P(H|\text{b}iased\ coin) = 3/4,$$

so

$$\begin{aligned} P(H) &= (1)(1/3) + (1/2)(1/3) + (3/4)(1/3) \\ &= 1/3 + 1/6 + 1/4 = 3/4. \end{aligned}$$

Therefore,

$$P(2H\ coin|H) = \frac{P(H|2H\ coin)P(2H\ coin)}{P(H)} = \frac{1/3}{3/4} = 4/9.$$

$$P(\mathrm{b}iased\ coin|H) = \frac{P(H|\mathrm{b}iased\ coin)P(\mathrm{b}iased\ coin)}{P(H)} = \frac{1/4}{3/4} = 1/3.$$

$$P(\mathrm{f}air\ coin|H) = \frac{P(H|\mathrm{f}air\ coin)P(\mathrm{f}air\ coin)}{P(H)} = \frac{1/6}{3/4} = 2/9.$$

Homework Assignment 2

Due date: Feb. 5, 2018.

Problem set on pages 87-95, 10th edition (pages 80-90, 11th edition)

1. Text, p. 87: 11
2. Text, p. 87: 13
3. Text, p. 88: 25
4. Text, p. 88: 26
5. Text, p. 90: 37
6. Text, p. 95: 79 (in 11th Edition this is 83)
7. Let $X_1, \cdots, X_n$ be independent r.v.'s each with the same exponential distribution with rate λ. Let $Y = \min(X_1, \cdots, X_n)$. Show that Y has an exponential distribution with rate $n\lambda$.
8. Let X have a Poisson distribution with mean λ. Find $P(X \text{ is odd})$. **Hint**: express

   $$h(\lambda) = P(X \text{ is odd})$$

   as a series in λ, obtain

   $$h'(\lambda) = \frac{\partial}{\partial \lambda} h(\lambda),$$

   show that

   $$\lim_{\lambda \to 0} h(\lambda) = 0,$$

   and assume that $h(\lambda)$ has the form

   $$h(\lambda) = ae^{b\lambda} + c,$$

   Then solve for the unknowns, a, b, c.

Solutions for Homework Assignment 2

1. (P. 87, 11) The draws are independent since the balls are replaced after each draw. So number of white balls in 4 draws has Binomial(4,.5) distribution.

$$P(X = 2) = \binom{4}{2}(.5)^4 = 3/8.$$

2. (P. 87, 13) Under the assumption of no ESP, probability of correct prediction of toss is 0.5, so expected number of correct predictions would be 5. Getting 7 correct is higher than expected and so is some evidence that there may be some ESP. Getting more than 7 correct would be even stronger evidence, so the relevant event to judge the strength of evidence for ESP is the event: getting at least 7 correct. Under the assumption of no ESP, the number correct has Binomial(10,.5) distribution.

$$\begin{aligned} P(X \geq 7) &= \sum_{k=7}^{10} \binom{10}{k}(.5)^{10} \\ &= \frac{176}{1024} = \frac{11}{64} = 0.172. \end{aligned}$$

3. (P. 88, 25) Let N denote the number of games played. Then the event that 7 games are played is the event that each team wins 3 games after 6 games are played. This probability is the Binomial probability of 3 successes out of 6 trials, and so is given by

$$\binom{6}{3}p^3(1-p)^3 = 20[p(1-p)]^3.$$

The function $h(p) = p(1-p)$, $0 < p < 1$, is maximized at the solution to $1 - 2p = 0$ (solution to derivative of h equals 0), which has solution $p = 1/2$. This is the maximum since 2^{nd} derivative of h is -2.

4. (P. 88, 26) For $i = 2$,

$$E(N) = 2P(N = 2) + 3P(N = 3) = 2(p^2 + (1-p)^2) + 3(2p(1-p)) = 2 + 2p(1-p).$$

For $i = 3$,

$$\begin{aligned} E(N) &= 3P(N = 3) + 4P(N = 4) + 5P(N = 5) \\ &= 3(p^3 + (1-p)^3) + 4(3p^3(1-p) + 3p(1-p)^3) + 5(6p^2(1-p)^2) \\ &= 3 + 3p(1-p) + 6p^2(1-p)^2. \end{aligned}$$

Since both are non-decreasing functions of $p(1-p)$, then they are maximized at $p = 1/2$ as shown in the previous problem.

5. (P. 90, 37) For $0 \le x \le 1$,

$$\begin{aligned} P(\max(X_1, \cdots, X_n) \le x) &= P(X_1 \le x, \cdots, X_n \le x) \\ &= \prod_{k=1}^{n} P(X_k \le x) \\ &= x^n. \end{aligned}$$

So, $F_M(x) = x^n$, $0 \le x \le 1$. Density function is the derivative,

$$f_M(x) = nx^{n-1}, \quad 0 \le x \le 1.$$

6. (P. 95, 79) We assume conditions hold under which the derivative can be passed inside the expectation. Then

$$K'(t) = \frac{d}{dt} \log(E[e^{tX}]) = \frac{E[Xe^{tX}]}{E[e^{tx}]}$$

and so

$$K'(0) = \frac{E(X)}{E(1)} = E(X).$$

Also,

$$\begin{aligned} K''(t) &= \frac{d}{dt} \frac{E[Xe^{tX}]}{E[e^{tx}]} \\ &= \frac{E[e^{tX}]E[X^2e^{tX}] - E[Xe^{tX}]E[Xe^{tX}]}{(E[e^{tX}])^2}. \end{aligned}$$

and so

$$K''(0) = \frac{E[X^2]E[1] - (E[X])^2}{(E[1])^2} = E[X^2] - (E[X])^2 = \mathrm{Var}(X).$$

7. (Extra)

$$\begin{aligned} P(Y > y) &= P(\min(X_1, \cdots, X_n) > y) \\ &= P(X_1 > y, \cdots, X_n > y) \\ &= \prod_{k=1}^{n} P(X_k > y) \\ &= e^{-ny\lambda}. \end{aligned}$$

So, $F_Y(y) = 1 - e^{-n\lambda y}$, which is d.f. of exponential distribution with rate $n\lambda$. Density is the dervative,

$$f_Y(y) = n\lambda e^{-n\lambda y}, \ y \ge 0.$$

8. (Extra 2) First note that

$$h(\lambda) = P(X \text{ is odd}) = \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} e^{-\lambda},$$

and

$$P(X \text{ is even}) = 1 - h(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} e^{-\lambda}.$$

Also,

$$\begin{aligned} h'(\lambda) &= \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} e^{-\lambda} - h(\lambda) \\ &= P(X \text{ is even}) - h(\lambda) \\ &= 1 - 2h(\lambda). \end{aligned}$$

This gives the differential equation,

$$h'(\lambda) + 2h(\lambda) = 1,$$

with boundary condition, $h(0) = 0$. Substituting the general solution,

$$h(\lambda) = ae^{b\lambda} + c,$$

gives

$$\begin{aligned} abe^{b\lambda} + 2ae^{b\lambda} + 2c &= 1 \\ a + c &= 0. \end{aligned}$$

The solution is

$$c = \frac{1}{2}, \ a = -\frac{1}{2}, \ b = -2.$$

So,

$$P(X \text{ is odd}) = \frac{1}{2}(1 - e^{-2\lambda}).$$

Alternatively, note that

$$\begin{aligned}
P(X \text{ is odd}) &= P(X \text{ is odd}) - P(X \text{ is even}) + P(X \text{ is even}) \\
&= \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} e^{-\lambda} - \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} e^{-\lambda} + 1 - P(X \text{ is odd}) \\
&= -e^{-\lambda} \sum_{k=0}^{\infty} \frac{(-\lambda)^{2k+1}}{(2k+1)!} - e^{-\lambda} \sum_{k=0}^{\infty} \frac{(-\lambda)^{2k}}{(2k)!} + 1 - P(X \text{ is odd}) \\
&= -e^{-\lambda} \sum_{n=0}^{\infty} \frac{(-\lambda)^{n}}{n!} + 1 - P(X \text{ is odd}) \\
&= -e^{-2\lambda} + 1 - P(X \text{ is odd}).
\end{aligned}$$

Therefore,

$$2P(X \text{ is odd}) = 1 - e^{-2\lambda},$$

and so,

$$P(X \text{ is odd}) = \frac{1}{2}(1 - e^{-2\lambda}).$$

Homework Assignment 3

Due date: Feb. 19, 2018

Exercises for Chapter 3, p. 163 (11th edition), p. 173 (10th edition)

1. Problem 3
2. Problem 4
3. Problem 11
4. Problem 15
5. Problem 17
6. Problem 24 (**hint**: use first step analysis, that is, condition on the outcome of the first coin flip)
7. Problem 40

Solutions for Homework Assignment 3

1. (p.173, 3).

$$\begin{aligned} E(X|Y=1) &= \frac{1\cdot\frac{1}{9}+2\cdot\frac{1}{3}+3\cdot\frac{1}{9}}{\frac{1}{9}+\frac{1}{3}+\frac{1}{9}} \\ &= \frac{10/9}{5/9} \\ &= 2. \end{aligned}$$

$$\begin{aligned} E(X|Y=2) &= \frac{1\cdot\frac{1}{9}+2\cdot 0+3\cdot\frac{1}{18}}{\frac{1}{9}+0+\frac{1}{18}} \\ &= \frac{5/18}{3/18} \\ &= \frac{5}{3}. \end{aligned}$$

$$\begin{aligned} E(X|Y=3) &= \frac{1\cdot 0+2\cdot\frac{1}{6}+3\cdot\frac{1}{9}}{0+\frac{1}{6}+\frac{1}{9}} \\ &= \frac{12/18}{5/18} \\ &= 2.4. \end{aligned}$$

2. (p.173, 4). Not independent because, for example, $P(X=1,Y=3)=0$, but

$$P(X=1)=2/9,\ P(Y=3)=5/18,$$

and so

$$P(X=1)P(Y=3)=5/81\neq P(X=1,Y=3).$$

3. (p.174, 11).

$$\begin{aligned} f_Y(y) &= \int_{-y}^{y} f(x,y)dx \\ &= \frac{1}{8}e^{-y}\int_{-y}^{y}(y^2-x^2)dx \\ &= \frac{1}{8}e^{-y}[y^2x-\frac{1}{3}x^3]|_{-y}^{y} \\ &= \frac{1}{8}e^{-y}\frac{4}{3}y^3 \\ &= \frac{1}{6}y^3e^{-y}. \end{aligned}$$

This is $Gamma(4,1)$ density. So,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{\frac{1}{8}e^{-y}(y^2-x^2)}{\frac{1}{6}y^3e^{-y}} \\ &= \frac{3}{4}y^{-3}(y^2-x^2), \end{aligned}$$

for $|x| \le y,\ 0 < y < \infty$. Since this conditional density is symmetric about $x = 0$ for each $0 < y < \infty$, then $E(X|Y = y) = 0$.

4. (p.174, 15).

$$f_Y(y) = \int_0^y f(x,y)dx = e^{-y},$$

$0 < y < \infty$. This is *exponential*(1) density, so

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = y^{-1},$$

$0 < x < y, \quad 0 < y < \infty$. This is Uniform$(0, y)$, and so

$$E(X^2|Y = y) = y^{-1}\int_0^y x^2dx = \frac{1}{3}y^2.$$

5. (p.175, 17). The joint density of X, Y is given by

$$f_{X,Y}(i,y) = \frac{y^i}{i!}e^{-y}Cy^{s-1}e^{-\alpha y} = \frac{Cy^i}{i!}y^{i+s-1}e^{-(1+\alpha)y}.$$

Hence,

$$\begin{aligned} P(X = i) &= \int f_{X,Y}(i,y)dy \\ &= \frac{Cy^i}{i!}\frac{\Gamma(i+s)}{(1+\alpha)^{s+i}}, \end{aligned}$$

and so,

$$\begin{aligned} f_{Y|X}(y|i) &= \frac{f_{X,Y}(i,y)}{P(X = i)} \\ &= Dy^{i+s-1}e^{-(1+\alpha)y}, \end{aligned}$$

where D is a constant that does not depend on y and is the value that makes this conditional density integrate to 1. This implies that the conditional density is $Gamma(i + s, 1 + \alpha)$.

6. (p.176, 24). Let N_{ij} denote the number of trials required to obtain at least i heads and at least j tails in a sequence of i.i.d. Bernoulli trials. Let $X_1, X_2, \cdots$ denote the Bernoulli trials. Note that N_{10} has a geometric distribution with success probability p and N_{01} has a geometric distribution with success probability $1-p$.

   a) Condition on X_1:

$$E(N_{11}) = E(N_{11}|X_1 = H)P(X_1 = H) + E(N_{11}|X_1 = T)P(X_1 = T)$$

   Note that if $X_1 = H$ then $N_{11} = 1 + N^*_{01}$ where N^*_{01} is the number of trials among $X_2, X_3, \cdots$ until the first tail appears. Likewise, if $X_1 = T$ then $N_{11} = 1 + N^*_{10}$ where N^*_{10} is the number of trials among $X_2, X_3, \cdots$ until the first head appears. Therefore N^*_{01}, N^*_{10} are each independent of X_1 and so

$$\begin{aligned} E(N_{11}) &= E(1 + N^*_{01}|X_1 = H)P(X_1 = H) + E(1 + N^*_{10}|X_1 = T)P(X_1 = T) \\ &= E(1 + N^*_{01})p + E(1 + N^*_{10})(1-p) \\ &= p(1 + 1/(1-p)) + (1-p)(1 + 1/p) \\ &= \frac{1}{p} + \frac{p}{1-p} \\ &= \frac{1 - p + p^2}{p(1-p)}. \end{aligned}$$

   b) Let M_{11} denote the number of heads during N_{11} trials. Then, as above,

$$\begin{aligned} E(M_{11}) &= E(M_{11}|X_1 = H)P(X_1 = H) + E(M_{11}|X_1 = T)P(X_1 = T) \\ &= E(1 + N_{01} - 1)p + 1(1-p) \\ &= (1-p) + \frac{p}{1-p} \\ &= \frac{1 - p + p^2}{1-p} \end{aligned}$$

   c) Let L_{11} denote the number of tails during N_{11} trials. Then $L_{11} = N_{11} - M_{11}$, and so,

$$\begin{aligned} E(L_{11}) &= E(N_{11}) - E(M_{11}) \\ &= \frac{1 - p + p^2}{p}. \end{aligned}$$

   d) First note that $E(N_{20}) = 2/p$. Similar to part a) above,

$$\begin{aligned} E(N_{21}) &= E(N_{21}|X_1 = H)P(X_1 = H) + E(N_{21}|X_1 = T)P(X_1 = T) \\ &= E(1 + N^*_{11})p + E(1 + N^*_{20})(1-p) \\ &= p + \frac{1 - p + p^2}{1-p} + (1-p) + \frac{2(1-p)}{p} \\ &= \frac{p^2}{1-p} + \frac{2}{p}. \end{aligned}$$

7. (p.179, 40).

a) Each time the prisoner returns to the cell, he is unable to determine which door he selected last time he was in the cell. So each time he returns, he is faced with the same situation as before. Let $X_k,\ k \geq 1$ denote the door selected on the k^{th} attempt. Then freedom occurs the first time the prisoner selects door 3. Let T denote the time until freedom. Note that if $X_1 = 1$, then he travels for 2 days and then is back where he started. Let T^* denote the number of days until freedom after his first return to the cell. Then

$$\begin{aligned} E(T|X_1 = 1) &= E(T^* + 2) \\ E(T|X_1 = 2) &= E(T^* + 3) \\ E(T|X_1 = 3) &= 0 \end{aligned}$$

Let $E(T) = m$. Since T and T^* have the same distribution, then

$$\begin{aligned} E(T) = m &= E(E(T|X_1)) \\ &= E(T|X_1 = 1)p_1 + E(T|X_1 = 2)p_2 + E(T|X_1 = 3)p_3 \\ &= (E(T^*) + 2)p_1 + (E(T^*) + 3)p_2 + 0 \\ &= (m + 2)p_1 + (m + 3)p_2 \end{aligned}$$

This equation is solved by

$$E(T) = m = \frac{2p_1 + 3p_2}{1 - p_1 - p_2} = \frac{2p_1 + 3p_2}{p_3}.$$

For this problem it is easier to obtain the variance directly from

$$Var(T) = E(T^2) - (E(T))^2$$

We can obtain $r = E(T^2)$ as above.

$$\begin{aligned} E(T^2) &= E(E(T^2|X_1)) \\ &= E(T^2|X_1 = 1)p_1 + E(T^2|X_1 = 2)p_2 + E(T^2|X_1 = 3)p_3 \\ &= E[(T^* + 2)^2])p_1 + E[(T^* + 3)^2]p_2 + 0 \\ &= p_1[E(T^2) + 4E(T) + 4] + p_2[E(T^2) + 6E(T) + 9] \end{aligned}$$

This gives

$$r = p_1 r + p_2 r + m(4p_1 + 6p_2) + 4p_1 + 9p_2,$$

and so

$$r = 2m^2 + \frac{4p_1 + 9p_2}{p_3},$$

Finally,

$$\mathrm{Var}(T) = r - m^2 = m^2 + \frac{4p_1 + 9p_2}{p_3}$$

For $p = (.5, .3, .2)$ we have

$$E(T) = 9.5, \quad \mathrm{Var}(T) = 113.75$$

b) Since the prisoner knows which door he selected previously, we can just consider the possible door choices that lead to freedom. They are:

$$3,\ 13,\ 23,\ 123,\ 213$$

The total times for these choices are:

$$0,\ 2,\ 3,\ 5,\ 5$$

and the corresponding probabilities are:

$$1/3,\ (1/3)(1/2),\ (1/3)(1/2),\ (1/3)(1/2)(1),\ (1/3)(1/2)(1)$$

Therefore, the p.m.f of T is

$$P(T = 0) = 1/3,\ P(T = 2) = 1/6,\ P(T = 3) = 1/6,\ P(T = 5) = 1/3,$$

and so

$$E(T) = 0 + 2/6 + 3/6 + 5/3 = 2.5,$$

$$\mathrm{Var}(T) = 0 + 4/6 + 9/6 + 25/3 - 25/4 = 51/12 = 4.25.$$

Homework Assignment 4

Due date: Feb. 28, 2018

1. Text, p. 276: 6 (11th edition: p. 261)
2. Text, p. 277: 14 (11th edition: p. 262)
3. Text, p. 277: 16 (11th edition: p. 263)
4. Text, p. 278: 20 (11th edition: p. 263)
5. Text, p. 278: 21 (11th edition: p. 263)
6. Text, p. 278: 22 **Hint, continued**. Use *modulo 13* arithmetic for the states of Y_n (11th edition: p. 263)
7. A computer program consists of a sequence of addresses that must be fetched from one of three locations, local memory (RAM), cache memory, or virtual memory (swap). A simple memory model can be expressed as follows: if the current address is in RAM, then the next address will be in RAM, cache, or swap with probabilities $0.8, 0.15, 0.05$, respectively; if the current address is in cache, then the next address will be in RAM, cache, or swap with probabilities $0.05, 0.9, 0.05$, respectively; if the current address is in swap, then the next address will be in RAM, cache, or swap with probabilities $0.2, 0.3, 0.5$, respectively. Let X_n denote the location of the n^{th} address and assume that $X_n,\ n \geq 1$ is a Markov chain. Find the proportion of time spent in each memory location over a long period of time.
8. Consider a component that begins operation with 0 damage. Suppose that at the end of a period of operation, it has accumulated damage N_1, where

   $$P(N_1 = k) = pq^k,\ k \geq 0,$$

   $0 < p < 1$, and $q = 1 - p$. If $N_1 \geq m$, where $m > 0$ is some fixed integer, then the component is replaced with an identical spare so that the damage at the beginning of the next period of operation for the component in use would be 0. Otherwise, the component begins the next period of operation with damage N_1. Damage is cumulative. That is, damage that occurs during a period of operation is added to the damage the component had at the beginning of the period, with the understanding that if the cumulative damage is m or higher at the end of a period, the component is replaced. Let N_k denote the damage that occurs during period k and assume that $\{N_k,\ k \geq 1\}$ are i.i.d. r.v.'s having the same geometric distribution given for N_1. Let $X_n,\ n \geq 1$ denote the damage of the component in use at the beginning of period n, and note that

   $$\begin{aligned} X_{k+1} &= X_k + N_k,\ \text{if } X_k + N_k \leq m - 1, \\ &\quad 0,\ \text{otherwise.} \end{aligned}$$

(a) What is the transition probability matrix of this Markov chain?

(b) Find its stationary distribution.

(c) What is the mean time between visits to state 0?

Note that the states are $0, 1, \cdots, m-1$.

Solutions for Homework Assignment 4

1. (p. 276, 6). Proof is by induction. The result holds trivially for $n = 1$. Now suppose it holds for n. From the Chapman-Kolmogorov equations,

$$\begin{aligned}
P^{(n+1)} &= P^{(n)}P \\
&= \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^n & \frac{1}{2}-\frac{1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{1}{2}(2p-1)^n & \frac{1}{2}+\frac{1}{2}(2p-1)^n \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \\
&= \begin{bmatrix} \frac{p}{2}+\frac{p}{2}(2p-1)^n+\frac{1-p}{2}-\frac{1-p}{2}(2p-1)^n & \frac{1-p}{2}+\frac{1-p}{2}(2p-1)^n+\frac{p}{2}-\frac{p}{2}(2p-1)^n \\ \frac{p}{2}-\frac{p}{2}(2p-1)^n+\frac{1-p}{2}+\frac{1-p}{2}(2p-1)^n & \frac{1-p}{2}-\frac{1-p}{2}(2p-1)^n+\frac{p}{2}+\frac{p}{2}(2p-1)^n \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}+\frac{2p-1}{2}(2p-1)^n & \frac{1}{2}-\frac{2p-1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{2p-1}{2}(2p-1)^n & \frac{1}{2}+\frac{2p-1}{2}(2p-1)^n \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{n+1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{n+1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{n+1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{n+1} \end{bmatrix}
\end{aligned}$$

 So the result holds for all n by induction.

2. (p. 277, 14).

 a) All states communicate so they are recurrent.

 b) All states communicate so they are recurrent.

 c) States 1,3 communicate so they are recurrent. States 4,5 communicate so they are recurrent. There is a positive probability of making a transition from state 2 into state 1, which is recurrent, so state 2 must be transient.

 d) States 1,2 communicate so they are recurrent. State 3 is an absorbing state. There is a positive probability of making a transition from state 4 to state 3, so state 4 is a transient state. There is a positive probability of making a transition from state 5 to state 1, so state 5 is a transient state.

3. (p. 277, 16). Let $C(i)$ denote the recurrent class to which i belongs. Suppose there exists state $j \notin C(i)$. If $i \nrightarrow j$, then

$$P_{ij}^n = 0, \ \forall n \geq 1.$$

 Now suppose $i \to j$. We already have seen that if state i is recurrent and if $i \to j$, then state j must be recurrent and $i \leftrightarrow j$. This would imply $j \in C(i)$, a contradiction. Therefore, if state i and state j do not communicate, then state j cannot be accessible from state i, that is, $P_{ij}^n = 0$, $\forall n \geq 1$.

4. (p. 278, 20). Let $\underline{v}$ denote a vector of 1's and let P_k denote the k-th column of P. Then

$$\underline{v}P_k = \sum_{j=0}^{M} P_{jk} = 1 = v_k$$

since P is doubly stochastic. Therefore, $\underline{v}$ is a solution to $\underline{v} = \underline{v}P$ and so

$$\pi_k = \frac{1}{\sum_{j=0}^{m} 1} = \frac{1}{M+1}.$$

5. (p. 278, 21).

a) The TPM is

$$P = \begin{bmatrix} 1-3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1-3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1-3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1-3\alpha \end{bmatrix}.$$

Note that for $n = 1$,

$$P_{i,i}^n = 1 - 3\alpha = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n.$$

and for $i \neq j$, $n = 1$,

$$P_{i,j}^n = \alpha = \frac{1}{4}[1 - (1-4\alpha)^n].$$

Now assume these hold for n. Then

$$\begin{aligned} P_{i,i}^{n+1} &= P^n[i,]P[,i] \\ &= [\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n](1-3\alpha) + 3\alpha[1-(1-4\alpha)^n] \\ &= \frac{1}{4}(1-3\alpha) + \frac{3}{4}(1-3\alpha)(1-4\alpha)^n + \frac{1}{4}3\alpha - \frac{1}{4}3\alpha(1-4\alpha)^n \\ &= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n[1-3\alpha-\alpha] \\ &= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n+1}. \end{aligned}$$

Also, for $i \neq j$,

$$\begin{aligned} P_{i,j}^{n+1} &= P^n[i,]P[,j] \\ &= [\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n]\alpha + \frac{1}{4}[1-(1-4\alpha)^n](1-3\alpha) + 2\alpha\frac{1}{4}[1-(1-4\alpha)^n] \\ &= \frac{1}{4} + \frac{1}{4}(1-4\alpha)^n[3\alpha - 1 + 3\alpha - 2\alpha] \\ &= \frac{1}{4}[1-(1-4\alpha)^{n+1}]. \end{aligned}$$

So the result holds for all n by induction.

b) This is a doubly stochastic matrix, so from the previous problem, $\pi_i = 1/4$, $1 \le i \le 4$.

6. (p. 278, 22). Define a Markov chain by

$$X_n = Y_n \text{ (modulo 13)}.$$

Then the states of this MC are $0, 1, \cdots, 12$ and the event that the sum of the dice is a multiple of 13 corresponds to state 0 of this MC. Also, for $i = 0, 1, \cdots, 6$

$$P_{i,j} = P(X_{n+1} = j | X_n = i) = 1/6, \;\; j = i+1, \cdots, i+6,$$

and for $7 \le i \le 11$,

$$P_{i,j} = P(X_{n+1} = j | X_n = i) = 1/6, \;\; j = i+1, \cdots, 12, \;\; 0, \cdots, i-7.$$

$P_{i,j} = 0$ otherwise. Now, for each state j there are only 6 states from which j is accessible, and the corresponding transition probability is 1/6 for each of those states. This implies that the TPM is doubly stochastic and so $\pi_0 = 1/13$.

7. (Extra 1) The TPM for this MC is

$$P = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.05 & 0.9 & 0.05 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

To find the stationary distribution we must solve $v = vP$, $\sum v_i = 1$. This gives the system of equations,

$$\begin{aligned} v_1 &= 0.8v_1 + 0.05v_2 + 0.2v_3 \\ v_3 &= 0.05v_1 + 0.05v_2 + 0.5v_3 \\ 1 &= v_1 + v_2 + v_3 \end{aligned}$$

Simplification yields

$$\begin{aligned} 4v_1 &= v_2 + 4v_3 \\ 10v_3 &= v_1 + v_2 \end{aligned}$$

Solving for v_2 in the first equation and substituting that into the second gives

$$\begin{aligned} v_1 &= \frac{14}{5}v_3 \\ v_2 &= \frac{36}{5}v_3. \end{aligned}$$

Hence,

$$1 = \frac{14}{5}v_3 + \frac{36}{5}v_3 + v_3$$

which implies that

$$\pi_3 = \frac{5}{55} = 0.0909, \;\; \pi_2 = \frac{36}{55} = 0.6545, \;\; \pi_1 = \frac{14}{55} = 0.2545.$$

So over a long period of time about 25.5% of the addresses are in RAM, about 65.4% are in cache, and about 9.1% are in swap.

8. (Extra 2) First note that if $X_k = 0$, then $X_{k+1} = 0$ if $N_k = 0$ or $N_k \geq m$. Also,

$$P(N_k \geq m) = \sum_{r=m}^{\infty} pq^r = q^m.$$

So,

$$P(X_{k+1} = 0 | X_k = 0) = p + q^m.$$

For $1 \leq i \leq m-1$,

$$\begin{aligned} P(X_{k+1} = 0 | X_k = i) &= P(X_k + N_k \geq m | X_k = i) \\ &= P(N_k \geq m - i) \\ &= q^{m-i}. \end{aligned}$$

For $1 \leq j \leq m-1$,

$$P(X_{k+1} = j | X_k = 0) = P(N_k = j) = pq^j.$$

For $1 \leq i \leq j \leq m-1$,

$$P(X_{k+1} = j | X_k = i) = P(N_k = j - i) = pq^{j-i}.$$

Therefore, the TPM is

$$P = \begin{bmatrix} p + q^m & pq & pq^2 & pq^3 & \cdots & pq^{m-1} \\ q^{m-1} & p & pq & pq^2 & \cdots & pq^{m-2} \\ q^{m-2} & 0 & p & pq & \cdots & pq^{m-3} \\ q^{m-2} & 0 & 0 & p & \cdots & pq^{m-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q & 0 & 0 & 0 & \cdots & p \end{bmatrix}$$

To find the stationary distribution, we must solve $\underline{\pi} = \underline{\pi} P$. Ignore the first equation and start with π_1.

$$\pi_1 = pq\pi_0 + p\pi_1 \Rightarrow q\pi_1 = pq\pi_0 \Rightarrow \pi_1 = p\pi_0.$$

$$\pi_2 = pq^2\pi_0 + pq\pi_1 + p\pi_2 \Rightarrow q\pi_2 = pq^2\pi_0 + p^2q\pi_0 \Rightarrow \pi_2 = p\pi_0.$$

We can use induction to solve the remaining equations. Suppose $\pi_i = p\pi_0$, $1 \le i \le k$. The equation for π_{k+1} is

$$\begin{aligned}
\pi_{k+1} &= \sum_{j=0}^{k+1} pq^{k+1-j}\pi_j \\
&= p\pi_{k+1} + pq^{k+1}\pi_0 + \sum_{j=1}^{k} p^2 q^{k+1-j}\pi_0.
\end{aligned}$$

This gives

$$q\pi_{k+1} = pq^{k+1}\pi_0 + p^2\sum_{j=1}^{k} q^{k+1-j}\pi_0,$$

and so,

$$\begin{aligned}
\pi_{k+1} &= pq^k\pi_0 + p^2\pi_0\sum_{j=1}^{k} q^{k-j} \\
&= pq^k\pi_0 + p^2\pi_0\frac{1-q^k}{1-q} \\
&= pq^k\pi_0 + p\pi_0(1-q^k) \\
&= p\pi_0.
\end{aligned}$$

Now use the equation $\sum \pi_i = 1$ to obtain

$$1 = \sum_{k=0}^{m-1}\pi_k = \pi_0 + (m-1)p\pi_0 = \pi_0(1 + (m-1)p),$$

and so,

$$\pi_0 = \frac{1}{1+(m-1)p}, \quad \pi_k = \frac{p}{1+(m-1)p}, \ 1 \le k \le m-1.$$

The mean time to return to state 0 is

$$m_0 = \frac{1}{\pi_0} = 1 + (m-1)p.$$

Homework Assignment 5

Due date: April 2, 2018

1. Text, 25, p. 264 (11th edition)

2. Text, 36, p. 266 (11th edition)

3. Text, 40, p. 266 (11th edition)

4. Let P be the transition probability matrix of an irreducible, aperiodic Markov chain, let $\underline{\pi}$ denote the stationary distribution of this Markov chain, and let A denote a subset of states with complement A^C. Prove that

$$\sum_{i \in A} \sum_{j \in A^C} \pi_i P_{i,j} = \sum_{i \in A^C} \sum_{j \in A} \pi_i P_{i,j}$$

5. Suppose the transition probability matrix of a Markov chain is given by

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.7 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   (a) Classify the states of this Markov chain.

   (b) Obtain the stationary distribution of the recurrent states.

   (c) Obtain the mean times to absorption for the transient states.

   (d) Find

$$\lim_{n \to \infty} P^n.$$

6. Components produced sequentially at a manufacturing facility are graded as 1, 2, or 3. Grade 1 is best, grade 2 is acceptable but lesser quality, and grade 3 is unacceptable. If a component is grade 1, the probability it is followed by grade 1,2,3, respectively, is $0.98, 0.01, 0.01$. If a component is grade 2, these probabilities are $0.90, 0.08, 0.02$. If a component is grade 3, these probabilities are $0.80, 0.15, 0.05$. Let X_n, $n \geq 1$, denote the grade of the n^{th} component and assume that X_n is a Markov chain.

   (a) Suppose the first 10 components are all grade 1. What is the probability that the next three components will be grade 1?

   (b) What proportion of components produced over a long period of time are grade 1?

(c) What is the mean time between grade 3 components?

(d) Suppose the manufacturing process is stopped for inspection if two consecutive grade 3 components are produced. If a grade 1 component is produced, what is the probability that the next two components will be grade 3?

Review Problems

1. A communications center has one link that is busy 40% of the time. Suppose the center checks this link every 10 minutes to determine whether or not the link is free, and suppose whether or not it is free at one time is independent of whether or not it is free any other time.
   a) What is the probability that the link will be busy each of the first 5 times it is checked?
   b) What is the probability that it will take more than 10 checks to find 2 free times?
   c) Given that the link was free exactly 2 times during the first 10 checks, what is the probability that the second free time came on the 10^{th} check?
   d) Given that the link was free exactly 2 times during the first 10 checks, what is the expected check number of the second free time?

2. A commercial web site offers products for sale on its site. During the day, hits on this site occur as a Poisson process with mean 40/hour. 4% of hits result in a purchase. Of those hits that result in a purchase, the purchase amounts are i.i.d. r.v.'s with an Exponential(rate=0.02) distribution, and are independent of the arrival process.
   a) What is the expected value and s.d. of the total sales per hour?
   b) What is the mean time between hits that result in a purchase?
   c) What is the mean time between hits that result in a purchase amount greater than \$100?

3. Non-fatal accidents occur at a particular intersection according to a Poisson process with a mean rate of 1 per 10 days. Suppose that 60% of these accidents result in no personal injuries that require hospitalization, 20% result in 1 person who required hospitalization, 15% result in 2 people who are hospitalized, 3% result in 3 people who are hospitalized, and the rest result in 4 people who are hospitalized.
   a) What is the mean time between accidents that have at least 1 person hospitalized?
   b) What is the expected number of people hospitalized as a result of accidents at this intersection over a 30 day period?
   c) Given that the first 10 days of a 30 day period had 1 accident with no one hospitalized, and 1 accident with 2 people hospitalized, what is the expected number of people hospitalized over the 30 day period?

4. Suppose $N(t)$ is a Poisson process with intensity λ. Let T_k be the waiting time to the k^{th} arrival and let $s > 0$ be a fixed, positive real number. Find

$$\begin{aligned} G_s(t) &= E[N(t)|N(s) = 2] \\ H_s(t) &= E[N(t)|T_2 = s] \end{aligned}$$

   for all $t > 0$.