EE 6340: Introduction to Telecommunications Networks

Project 3

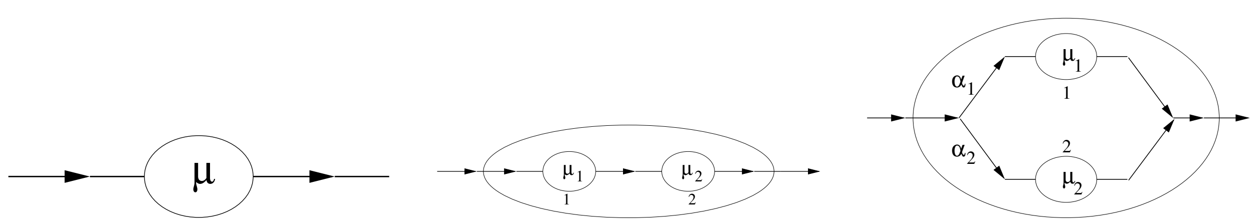
A. Study of the impact of the service time distribution.

A queuing system has a *single* service facility that can have one the following distributions:

- **Deterministic distribution**. Figure 1(a) shows an oval which represents the service facility with a single server. The service time of the single server is deterministic, equal to $\frac{1}{\mu}$.
- **Exponential distribution**. Figure 1(a) shows an oval which represents the service facility with a single server. In this case, the service time of the single server has an exponentially distributed pdf with average $\frac{1}{\mu}$.
- **Erlangian distribution**. Figure 1(b) shows a large oval which represents the service facility. Service consists of two stages in series, or tandem, i.e., each customer that enters service will undergo two stages of service. The two stages are performed sequentially. Only one customer is allowed in the oval at any time, i.e., a customer can begin service of the first stage only when no other customer is in service (first or second stage). Each server stage $i = 1, 2$ has an exponential distributed service time with average $\frac{1}{\mu_i}$. The two exponentially distributed service times are independent. This two-stage server facility works as follows. When both stages are empty, a customer can enter the first stage and will remain there for a random time with exponential distribution. Upon departure from the first stage, the customer proceeds immediately to the second stage and will remain there for a random time with exponential distribution. Only after being served at the second stage, the customer may depart, and a new customer may enter the service facility.

  The service time distribution of the service facility is referred to as *Erlangian distribution* and for a two-stage server, it is denoted as E_2.
- **Hyperexponential distribution**. Figure 1(c) shows a large oval which represents the service facility. The internal structure of the service facility is composed of two service stages in parallel. Each service stage $i = 1, 2$ has an exponential distributed service time with average $\frac{1}{\mu_i}$. The two exponentially distributed service times are independent. Only one customer is allowed in the oval at any time, i.e., a customer can begin service when no other customer is in service (first or second stage). This two-stage server facility works as follows. When both stages are empty, a customer can enter service. Only one of the two stages will be chosen by the customer: stage 1 with probability α_1, and stage 2 with probability α_2, where $\alpha_1 + \alpha_2 = 1$. Upon completion of service in the chosen stage, the customer departs, and a new customer can enter the service facility.

  The service time distribution of the service facility is referred to as *hyperexponential distribution* and for a two-stage server, it is denoted as H_2.



(a) The single-stage server (b) The two-stage series server (c) The two-stage parallel server

Figure 1: Service facilities

1. Let assume that the service rate for the Erlangian service facility is $\mu_1 = \mu_2 = 2\mu$. Demonstrate that the expected service time for service facilities with deterministic distribution ($\bar{X}_D$), exponential distribution ($\bar{X}_M$), and Erlangian distribution ($\bar{X}_{E_2}$) is equal to $\frac{1}{\mu}$.
2. Let assume that the service rate for the hyperexponential server facility is $\mu_1 = \mu/2$ and $\mu_2 = 2\mu$. Find the value of the probability α_1 that gives an expected service time ($\bar{X}_{H_2}$) equal to $\frac{1}{\mu}$.
3. Under the above conditions, evaluate the second moment of the service time for the different service facilities, i.e., $\bar{X}^2_D$, $\bar{X}^2_M$, $\bar{X}^2_{E_2}$, and $\bar{X}^2_{H_2}$.
4. Assume that the arrivals to any of the service facilities follow a Poisson process with rate λ and that the capacity of the queue is unlimited. **Plot #1:** plot the average waiting time for each one of the service facilities described above (i.e., W_D, W_M, W_{E_2}, and W_{H_2}), versus the utilization factor ρ, when $\mu = 1$.
5. Assume that in the Erlangian distribution the service rate of the stages are related as $\mu_2 = K\mu_1$, where $K \geq 0$ is a constant. What is the value of K that achieves the minimum average waiting time W_{E_2}, when the average service rate of the server facility is kept constant and equal to $\bar{X}_{E_2} = \frac{1}{\mu}$? What is the ratio of W_{E_2} and W_M? Can the Erlangian service facility have $W_{E_2} > W_M$? Provide an intuitive explanation of your answer.
6. Assume that in the hyperexponential distribution the service rate of the stages are related as $\mu_2 = K\mu_1$, where $K \geq 0$ is a constant. **Plot #2:** plot the second moment ($\bar{X}^2_{H_2}$) of the service facility service time for $\alpha_1 = 0.1, 0.5$, and 0.9, versus K when the expected facility service time is kept constant and equal to $\bar{X}_{H_2} = \frac{1}{\mu}$. In the same plot, add $\bar{X}^2_M$.
7. From plot #2, what are the values of K and α_1 that give the minimum $\bar{X}^2_{H_2}$? Can $\bar{X}^2_{H_2}$ be smaller than $\bar{X}^2_M$? If $\bar{X}^2_{H_2} \geq \bar{X}^2_M$ (or $\bar{X}^2_{H_2} \leq \bar{X}^2_M$), is it possible to conclude that $W_{H_2} \geq W_M$ (or $W_{H_2} \leq W_M$) holds always when $\bar{X}_{H_2} = \bar{X}_M$?