The Multiplexing Gain of Wireless Networks

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***Abstract*— At high SNR the capacity of a point-to point MIMO system with N_T transmit antenna and N_R receive antenna is $\min\{N_T, N_R\}\log(\mathbf{SNR}) + O(1)$. The factor in front of the $\log$ is called the multiplexing gain. In this paper we consider a network with $2N$ nodes (N source destination pairs) that each have only a single antenna. These single antenna nodes could cooperate to form larger virtual arrays, usually called cooperative diversity, user cooperation, or coded cooperation. The question we ask is: how large a multiplexing gain is possible. We prove that for $N = 2$ the multiplexing gain is 1, and consider generalizations to larger networks.**

I. Introduction

It is well known for that in a variety of channels, the rate transmitted over a channel has a tradeoff with the reliability of the practical transmission systems that are limited to a certain delay and/or computational complexity. This is characterized by the *coding exponent* of the channel.

Zheng and Tse posed a similar question regarding the optimal tradeoff between the reliability and the throughput of a MIMO communication system under quasi-static, flat Rayleigh fading. They observed that when the MIMO system has N_T transmit and N_R receive antennas, for any multiplexing factor $r < \min\{N_T, N_R\}$, the optimal diversity gain $d(r)$ is characterized by a piece-wise linear function on the (r, d) plane going through points $(k, (N_T - k)(N_R - k))$, $k = 0, \ldots, \min\{N_T, N_R\}$.

Cooperative communication is motivated by the idea that in a quasi-static channel, additional antennas of other mobiles can be used to generate a virtual MIMO system. Among the various works in this area one might name a few [8], [7].

Let us look at a prototypical example of a 2×2 user system. In this case, information flow is from two wireless nodes to another two wireless node. This, altogether, forms a traditional *interference channel*. However, the difference in our case is that the two transmitting users are allowed to listen to each other's transmissions and make decisions based on whatever they might infer (transmit-side cooperation). Similarly, the two receiving users are allowed also to exchange information (receive-side cooperation). See Figure 1.

Cooperative communication is in certain ways inspired by the idea of spatial diversity that also motivated MIMO systems. Thus it is only natural to ask whether the cooperative channel can support the same throughputs achieved by the MIMO channel. It is now well known [5], [9] that the high-SNR ergodic capacity of the MIMO channel follows $C \sim \min\{N_T, N_R\}\ \log(SNR) + O(1)$. For example, in a simple 2×2 MIMO system, throughput on the order of $2\log(SNR)$ are achievable.

The MIMO channel has certain advantages over the cooperative channel. In the MIMO systems, the code applied at the transmit side is simultaneously calculated and applied to the two antennas. Similarly, at the receive side, the decoding can utilize the signals from both receive antennas simultaneously. In the cooperative channel, the two transmit nodes do not have prior information about each other's data, and the receive channels do not know about each other's receive waveform. Any such information at the transmit or receive side must be exchanged over a wireless medium, which itself is potentially unreliable.[1]

It is then instructive to start from the standard MIMO channel, and make small adjustments towards the cooperative channel and see what happens. For example, if we remove the link between the two transmit antennas,

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[1]Another potential difference between the MIMO and cooperative systems is that in the former, transmission from multiple antennas is easily synchronized, which is critical to the performance of the MIMO system. In the cooperative case, some studies assume an asynchronous mode, and some studies assume synchronous mode. Furthermore, in the cooperative channel we have the question of full-duplex vs. half-duplex operation of the nodes. In this paper, we take the most liberal, permissive attitude towards cooperation (synchronized AND full-duplex) and show that even in that case it does not achieve the same multiplexing gain as the MIMO channel.

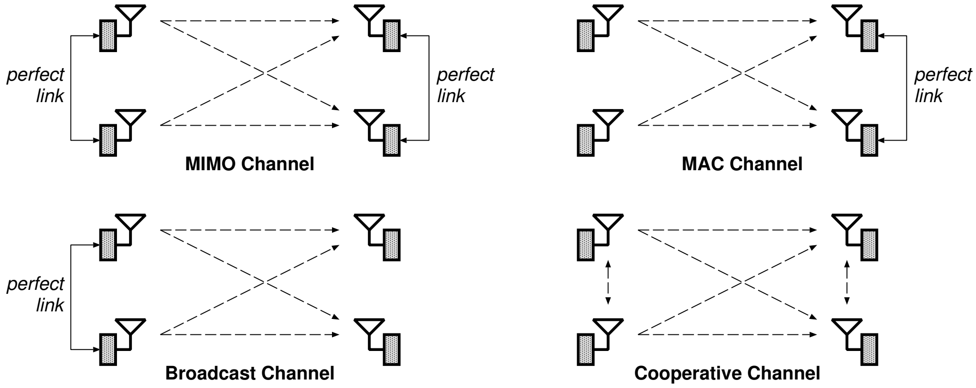

Fig. 1. Two by two systems

we will arrive at a two-user multiple access channel, where two single-antenna users communicate with a base station that has two antennas. Assuming both users have equal power, it is easy to see from [4] that the sum capacity of the system is on the order of $2\log(SNR)$.

If we start from the MIMO system, and sever the link between the two receivers, then we arrive at a broadcast channel, where a two-antenna transmitter communicates with two single-antenna receivers. Again it is known [2] that the sum capacity of this system is on the order of $2\log(SNR)$.

Finally, sever the reliable links at both transmit and receive side to arrive at a cooperative channel. The question: is the sum-rate capacity of $2\log(SNR)$ still supported by this system? The general message of this paper is that, unfortunately, this is impossible. In fact, the multiplexing gain of this system is no better than one. Thus, despite the exchange of information on the wireless links between transmitters and receivers, essentially the users have to share the channel between them. It is perhaps important to note that in terms of raw numbers the capacity of the system does indeed improve (sometimes significantly) by cooperation. However, in terms of the *multiplexing gain*, which is the factor in front of $\log(SNR)$, the system does not improve. It seems that the multiplexing gains promised by the MIMO systems are critically dependent on a tight coordination among the transmit antennas on the one side, or among the receive antennas on the other side; a level of coordination that seemingly cannot be achieved by the wireless connections available to cooperative communication.

II. System Model

We consider a network with $2N$ nodes. Node i, $i \leq N$ wants to transmit a message w_i to node $i+N$. The channel between nodes i and j is defined by a complex attenuation c_{ji}, and the received signal at node is subject to white Gaussian noise with power σ^2. By scaling we can transform this into an equivalent normalized channel with $\sigma^2 = 1$ and $c_{i+N,i} = 1$. The received signal at node j is

$$Y_j[n] \quad = \sum_{i=1, i\neq j}^{N} c_{j,i} X_i[n] + Z_j[n] \tag{1}$$

We make the following assumptions about the nodes

- all nodes have perfect knowledge about the channel, i.e., know all c_{ij}.
- all nodes are perfectly synchronized.
- all nodes are subject to a strict causality constraint: the output $X_i[n]$ only depends on past input, $Y_i[n-1], \ldots, Y_i[1]$
- Nodes operate in full-duplex mode

III. Theorem and Proof for 2 Pairs

Theorem 1: Consider a 2-pair network with full cooperation. If $|c_{41}| < 1$ the following bound applies

$$R_2 \quad \leq \quad \log\left(\frac{1+\left(|c_{41}|\sqrt{P_1}+\sqrt{P_2}+|c_{43}|\sqrt{P_3}\right)^2}{|c_{41}|^2 2^{R_1}\kappa + 1 - |c_{41}|^2}\right) \tag{2}$$

$$\kappa \quad = \quad \frac{(1+P_1)^2}{(1+(1+|c_{41}|^2)P_1)\,(1+(1+|c_{21}|^2)P_1)} \tag{3}$$

Furthermore, the resulting bound on $R_1 + R_2$ is an increasing function of R_1.

If $|c_{41}| > 1$ the following bound applies

$$\begin{aligned} &R_1 + R_2 \\ &\leq \log\left(1+\left(|c_{41}|\sqrt{P_1}+\sqrt{P_2}+|c_{43}|\sqrt{P_3}\right)^2\right) + \\ &\quad \log\left(\frac{\left(1+2|c_{41}|^2P_1\right)\left(1+(|c_{41}|^2+|c_{21}|^2)P_1\right)}{(1+|c_{41}|^2P_1)^2}\right) \end{aligned} \tag{4}$$

Corollary 1: The maximum multiplexing gain for a network with $N = 2$ pairs is 1.

Proof:

The bound is inspired by bounds for the Gaussian interference channel in [3], [6]. We modify the system as indicated on Figure 2. We will argue that each step either leads to an equivalent system or a system with a larger capacity region. In the first step we give node 3 as side-information the message w_2 and the signals received at nodes 2 and 4, which we will call Y_3' and Y_3''. With this node 3 can predict perfectly what node 2 and 4 will transmit, and these links can be removed. At the same time we replace the transmitter in node 3 with an extra node, node $4'$ that has access to message w_2 and the received signal Y_3'. Finally, to take care of feedback, we let nodes 1 and 2 know the past received signal at all other nodes. The argument for the last step in Figure 2 is the same as in Figure 6(b) to 6(c) in [3]: since $|c_{41}| < 1$ node 4 receives a degraded (lower SNR) version of the signal received at node 3. This is equivalent to taking the received signal at node 3 and adding additional Gaussian noise with power

$$E[|Z_4'|^2] = \frac{1-|c_{41}|^2}{|c_{41}|^2} \tag{5}$$

We consider upper bounds for the system in Figure 2(c).

Let $\mathbf{Y}_j = \{Y_j[n], Y_j[n-1], \dots, Y_j[1]\}$ and $\mathbf{Y}_j[i] = \{Y_j[i-1], Y_j[i-2], \dots, Y_j[1]\}$. We can bound R_1 as follows

$$\begin{aligned}
&nR_1 &(6)\\
&\leq H(\mathbf{Y}_3, \mathbf{Y}_3', \mathbf{Y}_3''|w_2) - H(\mathbf{Y}_3, \mathbf{Y}_3', \mathbf{Y}_3''|X_1, w_2)\\
&= \sum_{i=1}^{n} H\left(Y_3[i], Y_3'[i], Y_3'''[i]\right| \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i], w_2\Big)\\
&\quad - \sum_{i=1}^{n} H(Z_3[i]) + H(Z_3'[i]) + H(Z_3''[i]) &(7)\\
&= \sum_{i=1}^{n} H\left(Y_3'[i], Y_3''[i]\right| Y_3[i], \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i], w_2\Big)\\
&\quad + \sum_{i=1}^{n} H\left(Y_3[i]\right| \mathbf{Y}_3[i], \mathbf{Y}_3'[i], w_2\Big)\\
&\quad - \sum_{i=1}^{n} H(Z_3[i]) + H(Z_3'[i]) + H(Z_3''[i]) &(8)\\
&\leq \sum_{i=1}^{n} H\left(Y_3'[i]\right| Y_3[i]\Big) + \sum_{i=1}^{n} H\left(Y_3''[i]\right| Y_3[i]\Big)\\
&\quad + \sum_{i=1}^{n} H\left(Y_3[i]\right| \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i], w_2\Big)\\
&\quad - \sum_{i=1}^{n} H(Z_3[i]) + H(Z_3'[i]) + H(Z_3''[i]) &(9)
\end{aligned}$$

by the chain rule.Here

$$\begin{aligned}
H\left(Y_3'[i]\right| Y_3[i]\Big) &\leq \log(2\pi e) + \log E\left[\text{var}[Y_3'[i]|Y_3[i]]\right]\\
&= \log(2\pi e)\\
&\quad + \log(E[\text{var}[c_{41}X_1[i]|X_1[i] + Z_3[i]\\
&\quad + \text{var}[Z_3']) \qquad (10)
\end{aligned}$$

By Lemma 1 below

$$\begin{aligned}
E\left[\text{var}[c_{41}X_1[i]|X_1[i] + Z_3]\right] &\leq |c_{41}|^2 P_1 - \frac{(|c_{41}|P_1)^2}{P_1+1}\\
&= \frac{|c_{41}|^2 P_1}{1+P_1} \qquad (11)
\end{aligned}$$

We have a similar bound for $H\left(Y_3''[i]\right| Y_3[i]\Big)$. With this we get

$$\begin{aligned}
&\sum_{i=1}^{n} H\left(Y_3[i]\right| \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i], w_2\Big) \geq nR_1 +\\
&n\log\left(\frac{(1+P_1)^2}{(1+(1+|c_{41}|^2)P_1)(1+(1+|c_{21}|^2)P_1)}\right) \qquad (12)
\end{aligned}$$

Rate R_2 can be bounded by

$$nR_2 \leq H(\mathbf{Y}_4) - H(\mathbf{Y}_4|w_2) \tag{13}$$

We can bound the first term by

$$\begin{aligned}
&H(\mathbf{Y}_4) \leq n\log(2\pi e) + n\log\Bigg(1 + \frac{1-|c_{41}|^2}{|c_{41}|^2} +\\
&\left(\sqrt{P_1} + \frac{\sqrt{P_2}}{|c_{41}|} + \frac{|c_{43}|\sqrt{P_3}}{|c_{41}|^2}\right)^2\Bigg) \qquad (14)
\end{aligned}$$

The second term in (13) can be bounded by (see top of next page) where we have used that conditioning reduces entropy and that $X_2[i]$ $X_4'[i]$ only depends on w_2 and $(\mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i])$. We can now use the (conditional) entropy power inequality [1], [4]:

$$\begin{aligned}
&H(Y_3[i] + Z_4'[i]|w_2, \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i])\\
&\quad\geq \log(2\pi e) +\\
&\quad\quad \log\left(2^{H(Y_3[i]|w_2, \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i])} + \text{var}[Z_4'[i]]\right) \qquad (22)
\end{aligned}$$

Then

$$\begin{aligned}
&H(\mathbf{Y}_4|w_2)\\
&\quad\geq n\sum_{i=1}^{n}\frac{1}{n}H(Y_3[i] + Z_4'[i]|w_2, \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i])\\
&\quad\geq n\log(2\pi e) +\\
&\quad\quad n\sum_{i=1}^{n}\frac{1}{n}\log\left(2^{H(Y_3[i]|w_2, \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i])} + 1\right) \qquad (23)\\
&\quad\geq n\log(2\pi e) +\\
&\quad\quad n\log\left(2^{\sum_{i=1}^{n}\frac{1}{n}H(Y_3[i]|w_2, \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i])} + 1\right) \qquad (24)
\end{aligned}$$

$$
\begin{aligned}
H(\mathbf{Y}_4|w_2) &= \sum_{i=1}^{n} H(Y_4[i]|\mathbf{Y}_4[i], w_2) && (15)\\
&\geq \sum_{i=1}^{n} H\left(Y_3[i] + c_{41}^{-1}\left(X_2[i] + c_{43}X_4'[i]\right) + Z_4'[i]|\mathbf{Y}_4[i], w_2, X_2[i], X_4'[i]\right) && (16)\\
&\geq \sum_{i=1}^{n} H(Y_3[i] + Z_4'[i]|\mathbf{Y}_4[i], w_2, X_2[i], X_4'[i]) && (17)\\
&\geq \sum_{i=1}^{n} H(Y_3[i] + Z_4'[i]|\mathbf{Y}_4[i], w_2, X_2[i], X_4'[i], \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i]) && (18)\\
&= \sum_{i=1}^{n} H(Y_3[i] + Z_4'[i], \mathbf{Y}_4[i]|w_2, X_2[i], X_4'[i], \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i])\\
&\quad - \sum_{i=1}^{n} H(\mathbf{Y}_4[i]|w_2, X_2[i], X_4'[i], \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i]) && (19)\\
&= \sum_{i=1}^{n} H(Y_3[i] + Z_4'[i], \mathbf{Z}_4[i]|w_2, X_2[i], X_4'[i], \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i])\\
&\quad - \sum_{i=1}^{n} H(\mathbf{Z}_4[i]|w_2, X_2[i], X_4'[i], \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i]) && (20)\\
&= \sum_{i=1}^{n} H(Y_3[i] + Z_4'[i]|w_2, \mathbf{Y}_3[i], \mathbf{Y}_3'[i], \mathbf{Y}_3''[i]) && (21)
\end{aligned}
$$

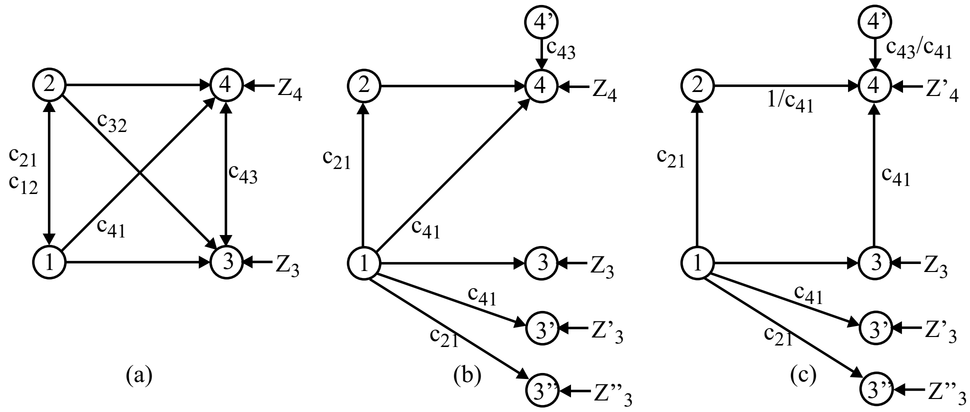

Fig. 2. Channel transformations used in the proof of theorem 1. All noise variables Z_i are independent, and all have power 1, except Z_4'' which has power $\frac{1-|c_{41}|^2}{|c_{41}|^2}$

where the last inequality is due to Jensen's inequality. Inserting (12) into this and using (14) we finally get

$$
R_2 \leq \log\left(\frac{1+\frac{1-|c_{41}|^2}{|c_{41}|^2}+\left(\sqrt{P_1}+\frac{\sqrt{P_2}}{|c_{41}|}+\frac{|c_{43}|\sqrt{P_3}}{|c_{41}|^2}\right)^2}{2^{R_1}\frac{(1+P_1)^2}{(1+(1+|c_{41}|^2)P_1)(1+(1+|c_{21}|^2)P_1)}+\frac{1-|c_{41}|^2}{|c_{41}|^2}}\right)
$$

which gives (2). It can be seen, for example by differentiation, that the sum $R_1 + R_2$ is an increasing function of R_1.

For $|c_{41}| > 1$ the transformation Figure 2(a-b) is still valid. Now, in Figure 2(b) we increase the gain on the link between nodes 1 and 3 from 1 to $|c_{41}|$, which can only increase rate. This can be transformed back to the normalized model by changing P_1 to $|c_{41}|^2 P_1$ and dividing the gain on all links out of node 1 with $|c_{41}|$. The transformation Figure 2(b-c) is then valid, and the bound (2) is valid with $|c_{41}| := 1$, $|c_{21}| := \frac{|c_{21}|}{|c_{41}|}$ and P_1

replaced with $|c_{41}|^2 P_1$, which gives

$$R_2 \quad \leq \quad \log \left(\frac{1 + \left(|c_{41}|\sqrt{P_1} + \sqrt{P_2} + |c_{43}|\sqrt{P_3}\right)^2}{2^{R_1} \frac{(1+|c_{41}|^2 P_1)^2}{(1+2|c_{41}|^2 P_1)(1+(|c_{41}|^2+|c_{21}|^2)P_1)}} \right)$$

The bound (4) on $R_1 + R_2$ is directly obtained from this. $\square$

Lemma 1: For any random variables X and Y with first and second order moments,

$$E\left[\mathrm{var}[Y|X]\right] \quad \leq \quad E[|Y|^2] - \frac{|E[XY]|^2}{E[|X|^2]} \tag{25}$$

Proof: First, notice that

$$E\left[\mathrm{var}[Y|X]\right] \quad = \quad E[Y^2] - E\left[|E[Y|X]|^2\right] \tag{26}$$

Second, the Cauchy-Schwartz inequality gives

$$|E[XY]| \quad = \quad |E[XE[Y|X]]| \tag{27}$$

$$\leq \quad \sqrt{E[|X|^2]}\sqrt{E\left[|E[Y|X]|^2\right]} \tag{28}$$

Inserting this gives (25). $\square$

IV. Generalization to more than 2 pairs

For a network with N pairs, we have the following statement

Proposition 1: The multiplexing gain for a network with N pairs is at most $\frac{N}{2}$.

The proposition is straightforward to prove, using the results in the previous section. Consider source nodes i and j. Suppose that all nodes in the network have side-information of all messages, except message i and j. The problem is then reduced to the 2-pair problem in the previous section. The remaining nodes can act as relays for nodes i and j, but it can be argued that relays cannot increase the multiplexing gain. Therefore

$$R_i + R_j \quad \leq \quad \log(\mathrm{S}NR) + O(1)$$

Adding up all such bounds give

$$(N-1)\sum_{i=1}^{N} R_i \quad \leq \quad \frac{N(N-1)}{2}\log(\mathrm{S}NR) + O(1)$$

The bound above shows that the multiplexing gain of a cooperative network does *not* grow in the same manner as MIMO structures, or in the same manner as the multi-access or broadcast channels mentioned in Section I.

We believe the bound obtained in Proposition 1 is not tight. The bound was obtained by adding $N-1$ inequalities whose construction required perfect knowledge by many nodes. Although at this time a proof is unavailable, we conjecture that the multiplexing gain for the general N-node network is one. Further work on this subject is ongoing.

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