Self-Characteristic Distributions

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Abstract

This paper investigates the fixed points of the characteristic function operator, i.e., we seek probability density functions whose characteristic functions are identical to themselves. A prominent example of a self-characteristic function is the Gaussian. Interestingly, however, the Gaussian is not unique in that respect. We present a formulation that systematically generates self-characteristic densities by applying a nonlinear transform to arbitrary positive semi-definite functions.

Keywords: Characteristic function, Fourier transform, fixed point, positive definite, Gaussian.

This paper addresses the question of probability density functions that have characteristic functions identical to themselves. We call such functions "self-characteristic." These functions constitute the fixed points of the characteristic function operator. It is noteworthy that this problem is similar to finding power spectra that map to autocorrelations of the same shape, since both operators are based on the inverse Fourier transform.

However, this problem does not reduce to the fixed point of the Fourier operator. The fixed points of the Fourier operator are easily obtained through its reciprocity property. Adopting the symmetric definition of Fourier transform:

$$\begin{aligned} F(\nu) &= \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\nu x} dx \\ f(x) &= \int_{-\infty}^{\infty} F(\nu)\, e^{j2\pi\nu x} d\nu \end{aligned} \tag{1}$$

we have the reciprocity relation $f(x) \overset{\mathcal{F}}{\to} F(\nu) \;\; \Rightarrow F(x) \overset{\mathcal{F}}{\to} f(-\nu)$. Therefore, for even $f(x)$, functions $F(x) + f(x)$ are fixed points of the Fourier transform. However, this process alone does not produce self-characteristic functions; $F(x) + f(x)$ may have an imaginary part, be negative in some interval(s), or may not have finite energy (in which case it cannot be normalized to integrate to one).

The best known example of a pdf with identical characteristic function is the Gaussian (with the appropriate variance). Due to many other interesting properties that the Gaussian enjoys alone among probability densities, one may be tempted to assert that the Gaussian is the only self-characteristic function. That, however, is not true. As a preview to the main result of this paper, we note one simple example of a non-Gaussian self-characteristic function[1]:

$$p(x) = sech(\pi x) = \frac{2}{e^{\pi x} + e^{-\pi x}} \quad . \tag{2}$$

As can be seen in Figure 1, the shape of this function is very similar to a Gaussian. The Gaussian and $sech(\pi x)$ are not the only self-characteristic functions. In the following, we present a systematic way of generating such functions.

Proposition 1 *Let $f(x)$ to be any square-integrable positive semi-definite function on $I\!R$, and take*

$$F(\nu) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\nu x} dx \quad . \tag{3}$$

[1] One of the reviewers brought it to the author's attention that W. Feller had knowledge of the existence of such pairs; in particular Feller pointed out the sech function in [2] (p. 503).

Furthermore, let $*$ denote the convolution operator

$$(f_1 * f_2)(x) = \int_{-\infty}^{\infty} f_1(\tau)\ f_2(x-\tau) d\tau$$

Then, the function

$$p(x) = f^2(x) + (F * F)\ (x) \quad , \tag{4}$$

with the proper normalization constant, is a self-characteristic density.

Proof:

$f(x) \in L_2$, hence $f^2 \in L_1$, $F(\nu) \in L_2$, and $F * F \in L_2$. Thus $p(x)$ is absolutely integrable and, with proper normalization, is a density. Noting the combination of reciprocity and convolution property of the Fourier transform, $f^2(x) \xrightarrow{\mathcal{F}} (F * F)(\nu)$ and $(F * F)(x) \xrightarrow{\mathcal{F}} f^2(-\nu) = f^2(\nu)$ (real, positive-semi-definite functions are even a.e.). It immediately follows that $p(x) \xrightarrow{\mathcal{F}} p(\nu)$. $p(x)$ is non-negative because $F(\nu)$, the Fourier transform of positive semi-definite $f(x)$, is non-negative, and the convolution of two non-negative functions is non-negative.

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Any self-characteristic function can be expanded into an expression similar to (4).

Proposition 2 *For any self-characteristic density function $p(x)$, there exists a square-integrable function $g(x)$ with Fourier transform $G(\nu)$ such that $p(x)$ can be expanded as*

$$p(x) = g^2(x) + (G * G)(x) \quad . \tag{5}$$

In particular, the function $g(x) = \frac{1}{\sqrt{2}} p^{\frac{1}{2}}(\nu)$, the positive root of $p(x)$, is a candidate generating function for $p(x)$.

Proof:

Since $p(x) \in L_1$, $g(x) = \frac{1}{\sqrt{2}} p^{\frac{1}{2}}(x) \in L_2$, thus $g(x)$ exists. Furthermore,

$$\begin{aligned} \mathcal{F}\{p(x)\} &= \mathcal{F}\{2\ g^2(x)\} = 2\ (G * G)(\nu) \\ &= p(\nu) = 2\ g^2(\nu) \end{aligned}$$

Thus $p(x) = 2\ g^2(x) = 2\ G * G(x)$; Equation (5) immediately follows.

■

We conjecture that, given a self-characteristic density $p(x)$, the function $\frac{1}{\sqrt{2}}\sqrt{p(x)}$ is positive semi-definite. This is borne out in known examples, but currently we have no proof. The implication of this conjecture is that our construction for self-characteristic functions is complete. Interestingly, in the inverse proposition $G(\nu)$ turns out to be positive semi-definite, but the same is not directly evident for $g(x)$. Alternatively, one may seek an altogether different construction for $g(x)$ which would be positive semi-definite, since the representations above are not unique. To show that these representations are not unique, consider the following positive semi-definite function:

$$f(x) = \Lambda(x) = \begin{cases} 1+x & \text{if} \quad -1 < x \leq 0 \\ 1-x & \text{if} \quad 0 < x \leq 1 \\ 0 & \quad\text{otherwise} \end{cases} \tag{6}$$

The self-characteristic function generated by this function, through (4), is:

$$p(x) = \alpha(\Lambda^2(x) + sinc^2(x) * sinc^2(x)) \;\; , \tag{7}$$

where

$$sinc(x) \doteq \frac{sin(\pi x)}{\pi x} \; .$$

and where α is a normalization constant so that $p(x)$ integrates to unity. (Because of reciprocity, a simple way to calculate α is to set $p(0) = 1$.) Thus, we modify the generating function from $\Lambda(x)$ to $g_1(x) = \sqrt{\alpha}\Lambda(x)$. The function $p(x)$ is shown in Figure 2. It is easy to see that the $g_2(x) = \sqrt{\alpha}\left[sinc^2(x) * sinc^2(x)\right]^{0.5}$ also generates the same self-characteristic function. Furthermore, the expansion given in Proposition 2 is:

$$g_3(x) = \sqrt{\frac{\alpha}{2}}\left[\Lambda^2(x) + sinc^2(x) * sinc^2(x)\right]^{\frac{1}{2}} \tag{8}$$

which is a generating function different from both g_1 and g_2 (see Figure 3). In fact, all functions

$$g_\theta(x) = \sqrt{\frac{\alpha}{2}}\left[(1-\theta)\Lambda^2(x) + (1+\theta)sinc^2(x) * sinc^2(x)\right]^{\frac{1}{2}} \qquad \theta \in [-1, 1]$$

generate the same self-characteristic function. The equivalence class of functions that generate a self-characteristic function are in general the solutions to

$$\mathcal{T}\{g(x)\} = [1 + \mathcal{F}]g^2(x) = p(x) \tag{9}$$

In conclusion, two notes:

- For convenience and elegance of expression, we used the symmetric definition of Fourier transform, which is based on linear frequency ν. The more common definition of characteristic function uses the angular frequency ω, where for an arbitrary probability density $p(x)$,

$$\Phi(\omega) = \int_{-\infty}^{\infty} p(x) e^{j\omega x} dx \quad . \tag{10}$$

  With this definition, fixed points exist only up to a multiplicative constant, due to the factor $\frac{1}{2\pi}$ when Parseval's identity is written in the angular frequency ω. If $p(x) \xrightarrow{\mathcal{F}} P(\nu)$ according to (1), then $\Phi(\omega) = P(\frac{-\omega}{2\pi})$. Results in this paper directly translate to the ω domain by a change of variable and appropriate normalization.

- By a simple property of Fourier transform, smoothness at origin and rate of decay are related in self-characteristic functions.

$$\int_{-\infty}^{\infty} x^n p(x)\, dx = \frac{d^n}{dx^n} p(x)|_{x=0} \quad . \tag{11}$$

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References

[1] R. N. Bracewell, *The Fourier Transform and its Applications*, McGraw-Hill, 1978.

[2] W. Feller, An introduction to Probability Theory and its Applications, vol. 2, John Wiley, 1966.

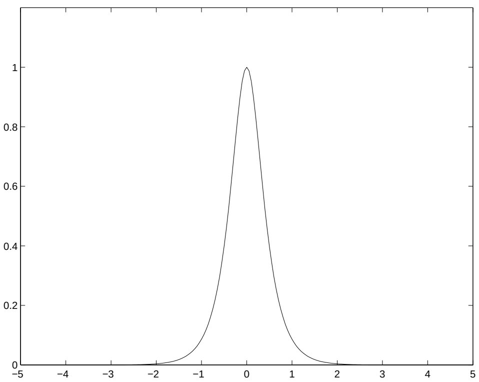

Figure 1: $sech(\pi x)$, self-characteristic

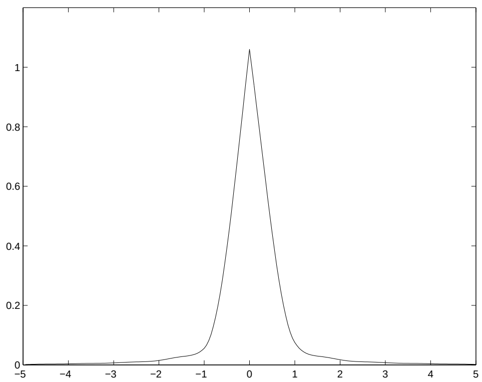

Figure 2: Self-characteristic function $\alpha(\Lambda^2(x) + sinc^2(x) * sinc^2(x))$

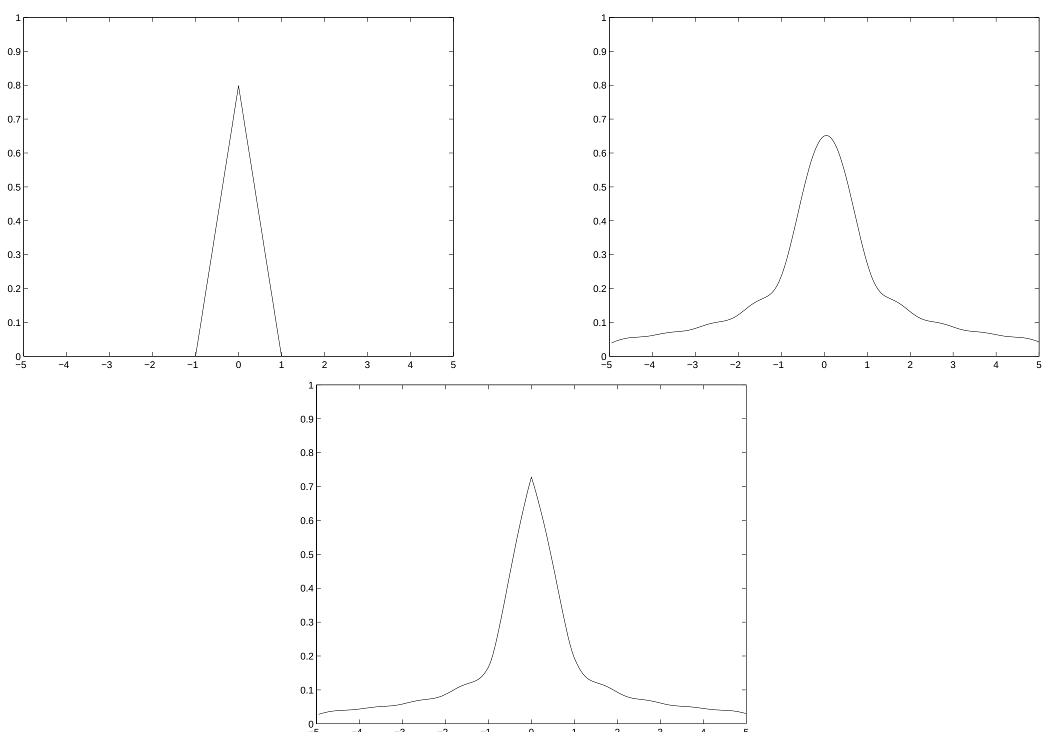

Figure 3: Different generating functions of $p(x) = \alpha(\Lambda^2(x) + sinc^2(x) sinc^2(x))$, shown in Figure 2. Top right: $g_1(x)$, top left: $g_2(x)$, bottom: $g_3(x)$ (see text).