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Data-enhanced Kalman filtering of colored process noise

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Abstract— **This paper studies the state estimation of nonlinear dynamical systems using stochastically forced linearized dynamics, where the stochastic input models the effect of process noise and the uncertainty caused by excluding nonlinear terms from the linearized model. The statistics of the input stochastic forcing greatly influence the design of estimation gains and can lead to the undesirable performance of state estimators. When the process noise is colored-in-time, conventional methods can fail to provide reasonable estimates of second-order statistics that are of interest in many feedback control applications. To address this problem, we utilize a recently developed framework for the dynamical modeling of input disturbances that provides statistical consistency at the level of second-order statistics with the underlying nonlinear dynamics. We demonstrate the efficacy of linear innovations models that result from this approach for the ensemble Kalman filtering of colored noise processes.**

Index Terms— **Covariance estimation, Kalman filter, optimal estimation, state estimation, stochastic dynamics, structured covariances.**

I. INTRODUCTION

We are interested in the continuous-time filtering problem in applications that involve complex nonlinear systems such as turbulent flows. The behavior of such systems is typically governed by partial differential equations that can be prohibitively expensive to simulate in real-time and whose lower complexity approximations give rise to substantial modeling uncertainties that hinder the utility of traditional Kalman filtering techniques [1] and their nonlinear extensions, e.g., [2]. For high-dimensional systems, the Ensemble Kalman Filter (EnKF) combines ideas from Monte Carlo techniques with Kalman-like methods for sequential data assimilation to provide a modern stochastic alternative for state estimation [3]–[5]. In this method, estimates of the mean state and its covariance are approximated by a judiciously designed random dynamical system of interacting particles (i.e., stochastic realizations that constitute an ensemble) that are propagated forward in time. The EnKF for linear Gaussian systems asymptotically converges to the Kalman result as the number of ensemble members becomes sufficiently large. Nevertheless, in practice, the propagation of a relatively small number of ensemble members has been shown to reasonably capture the dominant directions of uncertainty in the estimation error.

Kalman filtering relies on an accurate statistical description of sources of uncertainty entering the process and output measurements. In most engineering applications, e.g., feedback control of turbulent flows or industrial process control setups, the covariances of disturbances that enter the process are unknown. Nevertheless, the statistics of such disturbances greatly influence the design of estimation gains and can lead to the undesirable performance of state estimators [6]–[8]. To address this challenge, estimation of input statistics from open-loop data has long been a subject of interest in the fields of adaptive filtering [9]–[15] and unknown input filtering [16]–[21]. Such approaches enable the realization of potentially colored-in-time Gaussian input processes via covariance factorization schemes [22], [23].

In this paper, we use the stochastic modeling framework developed in [24]–[26] together with the stochastic EnKF to address the estimation of high-dimensional systems with unknown stochastic process noise. The unknown disturbances are assumed to be wide-sense stationary, but are not restricted to white-in-time processes. To explain the second-order statistics of a subset of noisy measurement variables we will use stochastically forced linearized models around a nominal trajectory of the nonlinear system or its long-time averaged state. Such models represent effective low-dimensional approximations that can preserve many essential qualitative features in many applications. In particular, it has been shown that the stochastically-forced linearized Navier-Stokes equations around the mean velocity profile qualitatively replicate the structural features of shear flows [27]–[32] and that suitable perturbations of linearized dynamics can reconcile their predictions with data from experimental measurements or numerical simulations [25], [26], [33], [34]. The stochastic modeling framework of [24]–[26] provides a systematic approach for identifying such dynamical perturbations using the solution of convex optimization problems. We use this approach to enhance the predictive capability of the EnKF to not only estimate deviations from the nominal trajectory of high-dimensional nonlinear systems, but to also provide reasonable estimates of second-order statistics. Such models are of great value in the feedback control of turbulent flows where the aim is not to simply track trajectories of the state, but to estimate and control second-order quantities that affect the kinetic energy of the flow and skin-friction drag.

The paper is organized as follows. In Section II, we formulate the continuous-time filtering problem using a variant of the EnKF. In Section III, we provide an overview of the stochastic modeling framework for the treatment of colored-in-time process noise. In Section IV, we provide details of the stochastic ensemble Kalman filtering algorithm. In Section V, we offer an illustrative example that utilizes the proposed approach. Finally, we provide concluding thoughts in Section VI.

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II. PROBLEM FORMULATION

Consider the general nonlinear system

$$\begin{array}{rcl}\dot{\tilde{x}} & = & f(\tilde{x}, w)\\ \tilde{y} & = & h(\tilde{x}) \,+\, v\end{array} \qquad (1)$$

where $\tilde{x}(t) \in \mathbb{C}^n$ is the state vector, $\tilde{y}(t) \in \mathbb{C}^p$ is the output, $w(t) \in \mathbb{C}^m$ denotes zero-mean stationary stochastic process noise, and $v(t) \in \mathbb{C}^p$ denotes white zero-mean measurement noise with covariance $V \succ 0$. Given partial correlations between a limited number of output components and a limited number of measurements (entries of $\tilde{y}$), we are interested in constructing an estimator that not only tracks changes in the state $\tilde{x}(t)$, but provides statistical consistency (up to a second order) with the original nonlinear model (1).

The Kalman filter and its extensions follow recursive algorithms that correct the estimate of the state via a gain × innovation error update formula to account for the new information contained in the most recent measurement [1]. While the Kalman filtering algorithm provides the posterior model only in linear Gaussian settings, it is often used as an approximate algorithm even in more general nonlinear settings or in conjunction with linearized models. In the control-oriented modeling of many complex systems that are governed by nonlinear partial differential equations, a promising approach is to leverage the underlying physics in the form of a prior linear time-invariant (LTI) model that arises from first principles, e.g., linearization of the Navier-Stokes equations around a stable flow state such as the turbulent mean velocity. Based on this, we consider an idealized linear approximation of dynamics of fluctuations around a nominal trajectory of the nonlinear system $\tilde{x}_0(t)$ in the presence of the nominal noise profile $\tilde{w}_0(t)$, i.e., $\dot{\tilde{x}}_0 = f(\tilde{x}_0, \tilde{w}_0)$, of the form

$$\begin{array}{rcl}\dot{x} & = & A\,x \,+\, B\,\mathbf{f}\\ y & = & C\,x \,+\, v\end{array} \qquad (2)$$

Here, $x := \tilde{x} - \tilde{x}_0$, and state-space matrices $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{p\times n}$ are given by

$$A \,=\, \left.\frac{\partial f}{\partial x}\right|_{\tilde{x}_0}, \qquad C \,=\, \left.\frac{\partial h}{\partial x}\right|_{\tilde{x}_0}.$$

In system (2), $\mathbf{f}(t) \in \mathbb{C}^m$ and $B \in \mathbb{C}^{n\times m}$ represent an unknown stochastic input and its corresponding input channel. This additive source of stochastic excitation is used to model the effect of exogenous disturbances w and v in (1) and the uncertainty caused by excluding nonlinear terms from the linearized equations.

If the stochastic input $\mathbf{f}(t)$ is white-in-time with covariance $W \succ 0$, an estimate $\hat{x}(t)$ for the state fluctuations around $\tilde{x}_0(t)$ is provided by the conventional Kalman filter

$$\dot{\hat{x}} \;=\; A\,\hat{x} \,+\, B\,\mathbf{f} \,+\, K\,(\tilde{y} \,-\, C\,\hat{x}) \qquad (3a)$$

$$\dot{P} \;=\; A\,P \,+\, PA^* \,+\, B\,WB^* \,-\, P\,C^*V^{-1}C\,P \qquad (3b)$$

where

$$K \;:=\; P\,C^*V^{-1}. \qquad (3c)$$

Here, $P(t)$ provides an approximation to the covariance of the estimation error, i.e., $\mathbf{E}\left[(\tilde{x}(t) - \hat{x}(t))(\tilde{x}(t) - \hat{x}(t))^*\right]$, with $\mathbf{E}(\cdot)$ denoting the expected value.

The sequential self-correcting property of the Kalman filter through feedback gain K provides robustness to certain levels of uncertainty inherent in the idealized model or measurement data. The optimality of the filtering algorithm, however, relies on a precise knowledge of disturbance covariances W and V in Eqs. (3). We next address the challenge of Kalman filtering in the presence of colored-in-time process noise with unknown dynamics.

III. STOCHASTIC DYNAMICAL MODELING OF UNKNOWN DISTURBANCES

We are provided with an N-member ensemble of noisy output measurements $\{\tilde{y}^i(t)\}_{i=1}^N$ over a time frame that is long enough to ensure a statistically steady state[1]. Each member of the ensemble corresponds to a trajectory of nonlinear system (1) resulting from a single realization of independent stochastic processes w and v. Only a subset of entries in the output vector $\tilde{y}_j$ are made available. It is desired to provide a realization for the colored-in-time stochastic input $\mathbf{f}(t)$ and prescribe an input matrix $B \in \mathbb{C}^{n\times m}$ with $m \leq n$, so that the idealized model (2) accounts for second-order statistics that can be computed from the partial measurements of $\tilde{y}(t)$. In this *training* window, we follow the stochastic modeling framework developed in [24]–[26] to formulate a covariance completion problem that identifies the second-order statistics of stochastic input in system (2) in order to reconcile dynamics with the sampled correlations of the output. As a measure of model parsimony, the optimization aims for a minimal number of degrees of freedom that are directly affected by stochastic excitation.

A. Covariance completion

Given matrices A, C, the available output covariance entries $Y_{ij} = \lim_{t\to\infty} \mathbf{E}\left[\tilde{y}_i(t)\,\tilde{y}_j^*(t)\right]$[2], and the covariance of measurement noise V, convex optimization problem

$$\begin{array}{ll}\underset{X,Z}{\text{minimize}} & -\log\det(X) \,+\, \gamma\,\|Z\|_\star\\ \text{subject to} & AX + XA^* + Z \;=\; 0\\ & (CXC^* \,+\, V)_{ij} \;=\; Y_{ij}, \;\; (i,j) \in \mathcal{I}\end{array} \qquad (4)$$

seeks Hermitian matrices $X, Z \in \mathbb{C}^{n\times n}$ that minimize a composite objective function that provides a trade-off between the solution to the maximum-entropy problem and the complexity of the forcing model. Here, the logarithmic barrier function ensures the positive definiteness of the state covariance matrix $X = \lim_{t\to\infty} \mathbf{E}\left[x(t)x^*(t)\right]$ [35] and results in a maximum-entropy stochastic realization [36], $\gamma > 0$ is a regularization parameter that reflects the relative weight of the nuclear norm regularizer $\|Z\|_\star = \sum_i \sigma_i(Z)$, which restricts the rank of Z [37], [38] and thereby the complexity

[1]This is a strong condition that can be relaxed to time frames that surpass potential transient stages.

[2]The expectation involves averaging over time and ensemble members.

of the forcing model; see [24] for additional details and an efficient optimization algorithm for solving (4). The solution to problem (4) not only provides statistical consistency with the nonlinear dynamical system (1) in matching partially observed second-order statistics, but completes the under-sampled covariances in a way that is consistent with the hypothesis that fluctuations around the nominal trajectory $\tilde{x}_0$ are generated by linear system (2).

Remark 1: In problem (4), we have assumed the covariance of measurement noise (the matrix V) to be known. It is noteworthy, however, that this covariance can also be added as an unknown to the convex optimization problem by either including additional matrix inequality $0 \prec V \preceq \tilde{V}$ as an additional constraint, which requires knowledge of an upper bound $\tilde{V}$, or considering robust formulations of the problem that augment the objective of (4) with a penalty on the violation of the moment matching constraint $(CXC^*)_{ij} = Y_{ij}$. The latter approach would require the selection of an appropriate regularizer to balance the effect of various components of an objective function with three components. Either way, our numerical experiments demonstrate a noticeable improvement in predicting true covariances, especially when the measurement noise is large.

B. *Stochastic realization of input* $\mathbf{f}$

The solution of problem (4) can be used to construct finite-dimensional systems with white noise inputs that realize the colored-in-time stochastic input $\mathbf{f}$ via covariance factorization techniques [22]. A class of generically minimal linear filters that have the same number of degrees of freedom as the finite-dimensional approximation of system (2) are given by [24], [25]:

$$\begin{aligned} \dot{\phi} &= (A - BK_f)\,\phi + B\,d \\ \mathbf{f} &= -K_f\,\phi + d \end{aligned} \tag{5}$$

where $\phi(t) \in \mathbb{C}^n$ is the state of the filter, $d(t)$ is a zero-mean white-in-time stochastic process of covariance Ω, and

$$K_f = \Big(\frac{1}{2}\,\Omega\,B^* - H^*\Big)X^{-1}.$$

Here, matrices B and H correspond to the factorization $Z = BH^* + HB^*$. In [25], it has been further shown that the state-space representation corresponding to the cascade connection of systems (5) and (2), which has twice as many states as the spatial discretization of system (2), is not controllable and thus not minimal. Removal of uncontrollable states yields a minimal realization of the form

$$\begin{aligned} \dot{x} &= (A - BK_f)\,x + B\,d \\ y &= C\,x + v. \end{aligned} \tag{6}$$

As the input matrix B results from the factorization of the low-rank solution Z to problem (4), the minimal realization (6) provides an alternative interpretation of colored-in-time input $\mathbf{f}$ as a white-in-time excitation together with a low-rank dynamical modification to the dynamic matrix A [25].

IV. STOCHASTIC ENSEMBLE KALMAN FILTER

In this section, we provide details of a stochastic ensemble Kalman filtering algorithm [39], [40] that we customize based on the particular class of (linear) innovations model presented in Section III. This choice is motivated by our desire to match statistical signatures of the underlying dynamics using linear models and the large-scale nature of systems that model fluid flows. Ensemble Kalman filtering algorithms replace the propagation of the covariance matrix $P(t)$ in Eq. (3b) with the empirical N-member (particle) approximation

$$P^{(N)}(t) = \frac{1}{N-1}\sum_{i=1}^{N}\Big(\tilde{x}^i(t) - \hat{x}^{(N)}(t)\Big)\Big(\tilde{x}^i(t) - \hat{x}^{(N)}(t)\Big)^*$$

which is computed using the simultaneous propagation of N independent trajectories of the underlying dynamics and observer dynamics. Here, $\tilde{x}^i(t)$ is the ith trajectory of the nonlinear dynamical system (1) resulting from the ith realization of stochastic inputs $w(t)$ and $v(t)$, and $\hat{x}^{(N)}(t)$ is the ensemble average of N state estimate trajectories resulting from the observer dynamics. In the stochastic variant of the EnKF, the particles $\hat{x}^i(t)$ evolve according to

$$\dot{\hat{x}}^i = A_f\,\hat{x}^i + B\,d^i + K\left(\tilde{y}^i - C\,\hat{x}^i - v^i\right) \tag{7}$$

where $i = 1,\ldots N$, $A_f = A - BK_f$ (cf. (6)), $K = P^{(N)}C^*V^{-1}$, and $\hat{x}^i(t)$ is the ith trajectory of the estimator due to the ith realization of the white stochastic input $d(t)$. In the stochastic EnKF the zero-mean white stochastic perturbation $v^i(t)$ is introduced in the innovations error to achieve consistency for the variance update. In the limit of $N \to \infty$, the stochastic EnKF can be shown to converge to the analytic filtering solution (3) [39], [41].

In the stochastic differential equation (SDE) (7), sources of uncertainty $d(t)$ and $v(t)$ are considered to be white processes that are defined as derivatives of Wiener processes [42], i.e.,

$$d(t) := \frac{\mathrm{d}\tilde{d}(t)}{\mathrm{d}t}, \quad v(t) := \frac{\mathrm{d}\tilde{v}(t)}{\mathrm{d}t}.$$

Here, $\tilde{d}(t)$ and $\tilde{v}(t)$ are vector-valued zero-mean processes of variance $\Omega \succ 0$ and $V \succ 0$. Based on this, the SDE (7) can be rewritten as

$$\begin{aligned} \mathrm{d}\hat{x}^i &= A_f\,\hat{x}^i\mathrm{d}t + B\,\mathrm{d}\tilde{d}^i \\ &\quad + P^{(N)}C^*V^{-1}\left((\tilde{y}^i - C\,\hat{x}^i)\mathrm{d}t - \mathrm{d}\tilde{v}^i\right) \end{aligned} \tag{8}$$

Since the covariance matrix $P^{(N)}$ depends on the ensemble members $\hat{x}^i(t)$, the noise term $\mathrm{d}\tilde{v}^i$ in (7) is of multiplicative nature. Multiplicative noise is not generally well-defined and its treatment calls for the adoption of a suitable stochastic calculus (e.g., Itō [43] or Stratonovich [44]). Algorithms for the continuous-time treatment of this SDE are provided in [45]. Herein, we instead use the Itō interpretation together with the forward Euler method as a compatible discretization scheme for propagating the solution forward in time.

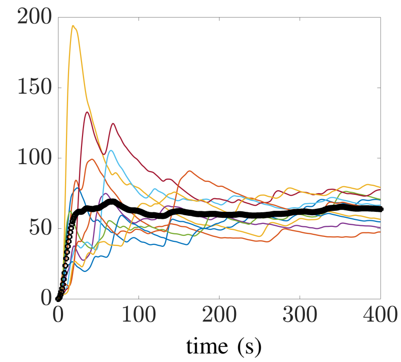

Fig. 1. Time evolution of the variance of the MSD system's state vector for 10 realizations of white-in-time forcing d to Eqs. (9). The variance averaged over the ensemble of all simulations if marked by the thick black line.

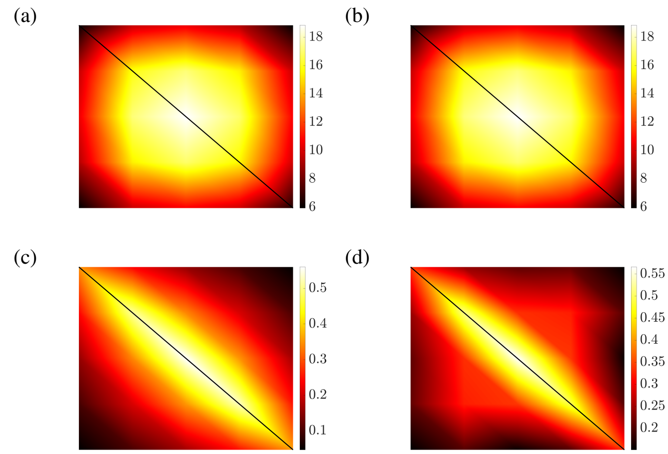

Fig. 2. Covariance of the position of 5 masses computed from a 10 particle ensemble of (a) noisy state measurements; and (b) state estimations. Covariance of the velocity of 5 masses computed from a 10 particle ensemble of (c) noisy state measurements; and (d) state estimations.

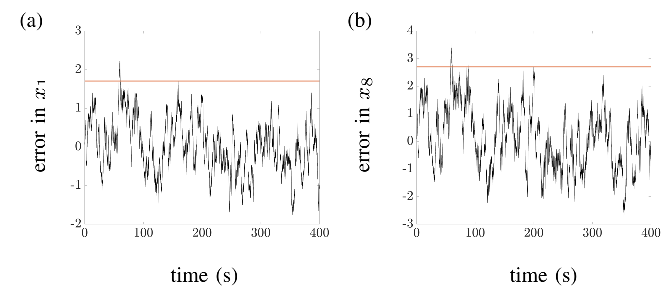

Fig. 3. State estimation errors for (a) $x_1(t)$ and (b) $x_8(t)$. The red lines denote the 3σ error bound.

V. AN EXAMPLE

We present an illustrative example to demonstrate the efficacy of our approach. Consider a mass-spring-damper system of 5 masses subject to stochastic disturbances that are generated by a low-pass filter,

$$\begin{aligned} \text{low-pass filter:} \quad \dot{\zeta} &= -0.1\,\zeta + d && \text{(9a)} \\ \text{MSD system:} \quad \dot{x} &= Ax + B\zeta && \\ y &= x + v && \text{(9b)} \end{aligned}$$

The state vector $x = [\,p^T \;\; v^T\,]^T$, contains the position and velocity of masses, d represents a zero-mean unit variance white process, and v represents zero-mean white measurement noise with covariance $V = 0.01I$. State and input matrices are

$$A = \left[\begin{array}{cc} O & I \\ -T & -I \end{array}\right], \qquad B = \left[\begin{array}{c} 0 \\ I \end{array}\right],$$

where O and I are zero and identity matrices and T is a symmetric tridiagonal Toeplitz matrix with 2 on the main diagonal and -1 on the first upper and lower sub-diagonals.

We collect an ensemble of state trajectories resulting from 10 linear stochastic simulations of the cascade connection of the low-pass filter (9a) and MSD system (9b); see Fig. 1. In these simulations, the low-pass filter is fed with white noise and it generates a colored-in-time input ζ to the MSD system. The empirical covariances of the position and velocity of masses that are computed from these noisy state measurements is shown in Figs. 2(a) and 2(c). The entries on the main diagonal of these covariances are used as data in the training stage to construct the idealized model (6) for state estimation. The solution of the covariance completion problem (4) guarantees that the modified dynamics (6) provide a completion of the sampled state covariance with a relative error of 21.6% when compared to the true state covariance. This is in spite of the fact that only entries on the main diagonal of the state covariance matrix where provided as data in problem (4).

We use the identified modified dynamics for the stochastic ensemble Kalman filtering algorithm presented in Section IV. The covariances that result from state estimate $\hat{x}(t)$ due to a 10-particle ensemble of simulation trajectories are presented in Figs. 2(b) and 2(d). The state covariance shows good recovery of the full state covariance matrix with a relative error of 20.7% with less than 0.1% error in matching the diagonal entries, i.e., one-point correlations of positions and velocities of the masses marked by the black lines in Fig. 2. We note that in the absence of the stochastic modeling procedure the conventional stochastic EnKF would result in more than 90% error in matching the state covariance matrix. Finally, the state estimation error is shown in Fig. 3 for two randomly chosen states. comparison with the 3σ bound shows reasonable performance of the data-enhanced stochastic EnKF in tracking the state.

VI. CONCLUDING REMARKS

We have demonstrated the efficacy of the stochastic modeling framework of [24]–[26] in providing effective innovations models of colored-in-time stochastic processes for the purpose of Kalman filtering. Given partially available correlations of noisy output measurements, we formulate convex optimization problems that identify the statistics of the unknown input forcing in order to reconcile dynamics with sampled correlations. The resulting innovations models are generically minimal in the sense that they have the same number of degrees of freedom as the finite-dimensional approximation of the linearized dynamics around a trajectory of the underlying nonlinear system. The augmented state Kalman filter is also guaranteed a minimal realization with the same number of states. We use these models to construct a stochastic ensemble Kalman filtering algorithm that not only estimates the state but also reasonably captures second-order statistics that contain physically relevant information. We use a small-size problem to demonstrate the ability of the data-enhanced EnKF in matching second-order statistics of the original system. Our efforts are directed at the efficient implementation of this estimation strategy for large-scale applications such as the feedback control of fluid flows.

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