COHERENCE THEORY (1)

- **Coherence** is the ability to form interference fringes
  - ▷ Fringe **visibility**:

$$\mathcal{V} := \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

  - ∘ $\mathcal{V}$ is dimensionless, and $0 \leq \mathcal{V} \leq 1$
  - ∘ For two coherent beams of equal intensity,

$$\mathcal{V} = 1$$

  - ∘ For two partially coherent beams of equal intensity, interfering with a path difference of $n\tau/c$,

$$\mathcal{V} = |\gamma_{12}(\tau)|$$

  - ∘ $\gamma_{12}(\tau)$ is the **complex degree of coherence**
  - ▷ Space-time propagation of $\gamma_{12}(\tau)$: Diffraction theory

COHERENCE THEORY (2)

- Michelson interferometer fringes
  - ▷ Assume no polarization effects ⇒ can use a scalar optical field u
  - ▷ Intensity at detector of recombined beams with time difference $\tau = 2d/c$:

$$\begin{aligned} I_d(\tau) &= \langle |K_1 u(t) + K_2 u(t+\tau)|^2 \rangle \\ &= K_1^2 \langle |u(t)|^2 \rangle + K_2^2 \langle |u(t+\tau)|^2 \rangle \\ &\quad + K_1\, K_2 \left[\langle u(t+\tau) u(t)^* \rangle + \langle u(t+\tau)^* u(t) \rangle \right] \\ &= (K_1^2 + K_2^2) I_0 + 2 K_1\, K_2 \, \mathrm{Re} \left[\Gamma(\tau) \right] \end{aligned}$$

  - ○ Average intensity: $I_0 := \langle |u(t)|^2 \rangle$
  - ○ **Self coherence function**: $\Gamma(\tau) := \langle u(t+\tau) u(t)^* \rangle$
    - ◇ For narrowband light (mean angular frequency $\overline{\omega}$),

$$\Gamma(\tau) \approx |\Gamma(\tau)| \, e^{-i[\overline{\omega}\tau - \alpha(\tau)]}$$

COHERENCE THEORY (3)

- Visibility of Michelson interferometer fringes
  - ▷ Mean intensity incident on the interferometer: $I_0 = \Gamma(0)$
  - ▷ Maximum and minimum intensities for time delays near τ are

$$I_{\max} = (K_1^2 + K_2^2) I_0 + 2\, K_1\, K_2\, |\Gamma(\tau)|, \quad I_{\min} = (K_1^2 + K_2^2) I_0 - 2\, K_1\, K_2\, |\Gamma(\tau)|$$

  - ▷ Fringe visibility with time difference $\tau = 2d/c$:

$$\mathcal{V}(\tau) = \frac{4\, K_1\, K_2\, |\Gamma(\tau)|}{2(K_1^2 + K_2^2) I_0} = \left(\frac{2 K_1 K_2}{K_1^2 + K_2^2} \right) \frac{|\Gamma(\tau)|}{I_0}$$

- **Complex degree of coherence**:

$$\boxed{\gamma(\tau) := \frac{\Gamma(\tau)}{\Gamma(0)}}$$

COHERENCE THEORY (4)

- Properties of the complex degree of coherence, $\gamma(\tau)$:
  - ▷ At zero time difference, $\gamma(0) = 1$
    - ∘ Fringe visibility depends on relative amplitudes of the two beams
  - ▷ As the time difference τ becomes infinite, $\gamma(\tau) \to 0$
    - ∘ Interference fringe visibility $\propto |\gamma(\tau)| \to 0$
- **Coherence time**:

$$\tau_{\mathrm{c}} := \int_{-\infty}^{\infty} |\gamma(\tau)|^2 \, d\tau$$

  - ▷ Intuitive meaning:
    - ∘ For time delays $\tau \ll \tau_{\mathrm{c}}$, the two beams in the Michelson interferometer are coherent (they form interference fringes)
    - ∘ For time delays $\tau \gg \tau_{\mathrm{c}}$, the beams are incoherent (no interference fringes; intensities add)

COHERENCE THEORY (5)

- Interference of light at two spatially separated points
  - ▷ Young's experiment: Intensity of light from two pinholes at P_1 and P_2 is measured at Q
  - ▷ Optical field at Q is

$$u(Q) = K_1 u\left(P_1, t - \frac{r_1}{c}\right) + K_2 u\left(P_2, t - \frac{r_2}{c}\right)$$

  - ▷ Intensity at $Q = I(Q) = \langle |u(Q)|^2 \rangle$:

$$\begin{aligned}
I(Q) &= \left\langle \left| K_1 u\left(P_1, t - \frac{r_1}{c}\right) + K_2 u\left(P_2, t - \frac{r_2}{c}\right) \right|^2 \right\rangle \\
&= |K_1|^2 \left\langle \left| u\left(P_1, t - \frac{r_1}{c}\right) \right|^2 \right\rangle + |K_2|^2 \left\langle \left| u\left(P_2, t - \frac{r_2}{c}\right) \right|^2 \right\rangle \\
&\quad + K_1 K_2^* \left\langle u\left(P_1, t - \frac{r_1}{c}\right) u\left(P_2, t - \frac{r_2}{c}\right)^* \right\rangle \\
&\quad + K_1^* K_2 \left\langle u\left(P_1, t - \frac{r_1}{c}\right)^* u\left(P_2, t - \frac{r_2}{c}\right) \right\rangle \\
&= |K_1|^2 I(P_1) + |K_2|^2 I(P_2) + 2\,\mathrm{Re}\left[K_1 K_2^* \Gamma_{12}((r_2 - r_1)/c)\right]
\end{aligned}$$

COHERENCE THEORY (6)

- Interference of light at two spatially separated points
  - ▷ Intensity at Q:

$$I(Q) = \underbrace{|K_1|^2 I(P_1) + |K_2|^2 I(P_2)}_{\text{incoherent sum of intensities}} + \underbrace{2\,\mathrm{Re}\,[K_1\, K_2^*\, \Gamma_{12}((r_2 - r_1)/c)]}_{\text{creates interference pattern}}$$

  - ▷ Wolf's **mutual coherence function**:

$$\boxed{\Gamma_{12}(\tau) := \langle u(P_1, t+\tau) u(P_2, t)^* \rangle}$$

  - ▷ Properties of $\Gamma_{12}(\tau)$:
    - ◦ Intensities at P_1 and P_2: $I(P_1) = \Gamma_{11}(0), \quad I(P_2) = \Gamma_{22}(0)$
    - ◦ As $\tau \to \infty$, $\Gamma_{12}(\tau) \to 0$
    - ◦ For narrowband light (mean angular frequency $\overline{\omega}$),

$$\Gamma_{12}(\tau) \approx |\Gamma_{12}(\tau)|\, e^{-i[\overline{\omega}\tau - \alpha_{12}(\tau)]}$$

COHERENCE THEORY (7)

- Interference of light at two spatially separated points
  - ▷ Intensity at Q:

$$I(Q) = |K_1|^2 I(P_1) + |K_2|^2 I(P_2) + 2\,\mathrm{Re}\,[K_1\, K_2^*\, \Gamma_{12}(\tau)]$$

  - ▷ Maximum and minimum intensities for time delays near τ are

$$I_{\max} = |K_1|^2 I(P_1) + |K_2|^2 I(P_2) + 2\,|K_1|\,|K_2|\,|\Gamma_{12}(\tau)|,$$

$$I_{\min} = |K_1|^2 I(P_1) + |K_2|^2 I(P_2) - 2\,|K_1|\,|K_2|\,|\Gamma_{12}(\tau)|$$

  - ▷ Fringe visibility in Young's experiment:

$$\mathcal{V}(\tau) = \frac{4\,|K_1|\,|K_2|\,|\Gamma_{12}(\tau)|}{2[K_1^2 I(P_1) + K_2^2 I(P_2)]}$$

$$= \left(\frac{2K_1\sqrt{I(P_1)}K_2\sqrt{I(P_1)}}{K_1^2 I(P_1) + K_2^2 I(P_2)}\right) \frac{|\Gamma_{12}(\tau)|}{\sqrt{I(P_1)I(P_2)}}$$

COHERENCE THEORY (8)

- **Complex degree of coherence** for light coming from two different points:

$$\gamma_{12}(\tau) := \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}}$$

  - ▷ Depends on the spatial positions of P_1 and P_2 as well as on $\tau = (r_2 - r_1)/c$
  - ▷ How does the mutual coherence function change as a wave propagates?
    - ∘ For a narrowband source,

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2; \tau) = \int_S \int_S \frac{\Gamma(\mathbf{r}'_1, \mathbf{r}'_2; \tau + (|\mathbf{r}_2 - \mathbf{r}'_2| - |\mathbf{r}_1 - \mathbf{r}'_1|)/c)}{\overline{\lambda}^2 |\mathbf{r}_2 - \mathbf{r}'_2||\mathbf{r}_1 - \mathbf{r}'_1|} \, d^2r'_1 \, d^2r'_2$$

    - ∘ This looks worse than it is...

COHERENCE THEORY (9)

- In order to separate the spatial and temporal coherence properties, we define the **mutual intensity**

$$J(\mathbf{r}_1, \mathbf{r}_2) := \Gamma(\mathbf{r}_1, \mathbf{r}_2; 0)$$

and the **complex coherence factor**

$$\mu(\mathbf{r}_1, \mathbf{r}_2) := \frac{J(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}}$$

- The **coherence area** is

$$A_{\mathrm{c}} := \int |\mu(\mathbf{r}, \mathbf{r} + \mathbf{r}')|^2 \, d^2 r'$$

COHERENCE THEORY (10)

- Propagation of the mutual intensity for narrowband light:

$$J(\mathbf{r}_1, \mathbf{r}_2) = \int_S \int_S \frac{J(\mathbf{r}'_1, \mathbf{r}'_2) e^{-i\bar{k}(|\mathbf{r}_2 - \mathbf{r}'_2| - |\mathbf{r}_1 - \mathbf{r}'_1|)}}{\bar{\lambda}^2 |\mathbf{r}_2 - \mathbf{r}'_2| |\mathbf{r}_1 - \mathbf{r}'_1|} \, d^2 r'_1 \, d^2 r'_2$$

▷ Example of a completely incoherent source:

$$J(\mathbf{r}'_1, \mathbf{r}'_2) = K \, I(\mathbf{r}'_1) \, \delta(\mathbf{r}'_1 - \mathbf{r}'_2)$$

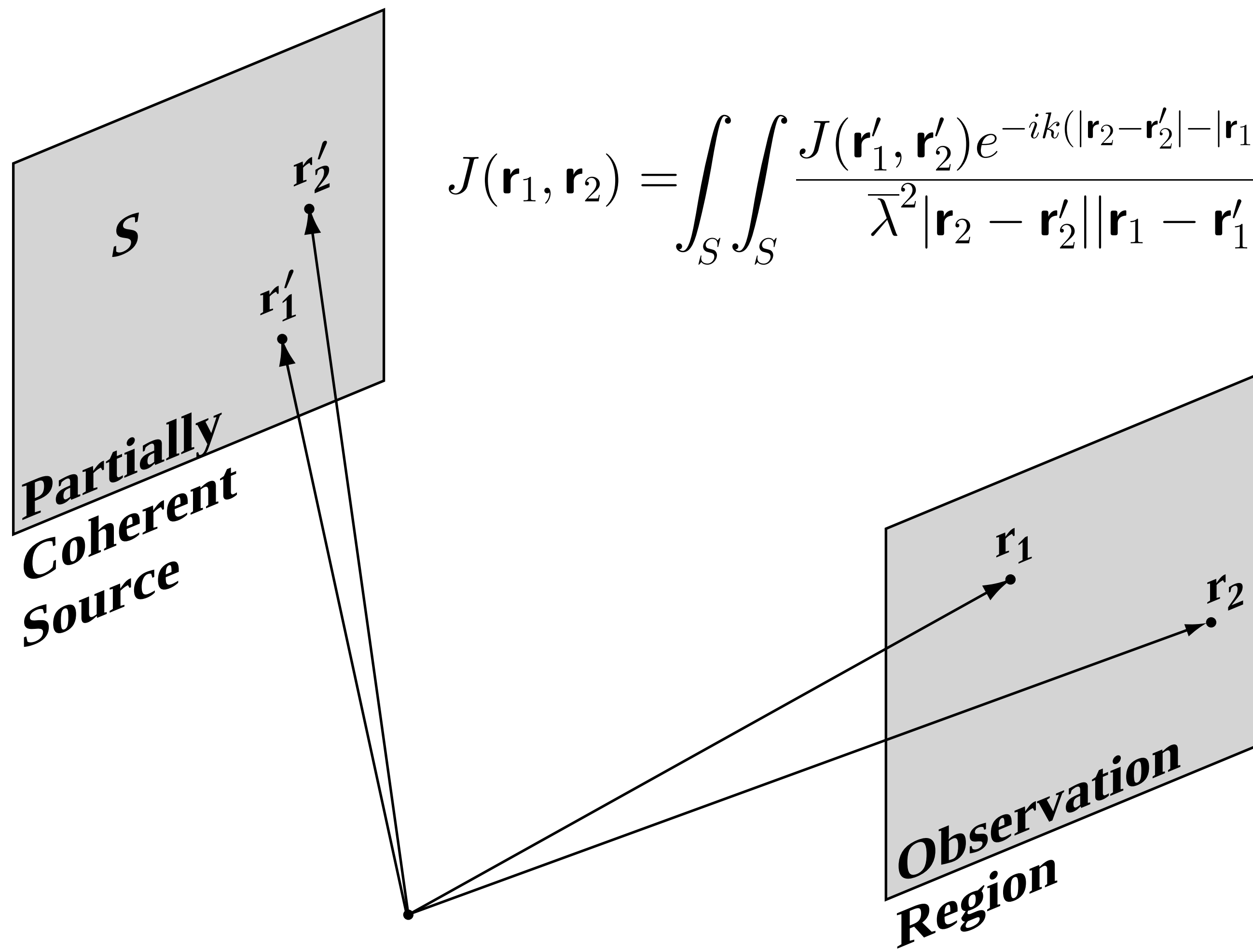
▷ Propagation of the mutual intensity from a narrowband incoherent source:

$$\begin{aligned} J(\mathbf{r}_1, \mathbf{r}_2) &= \int_S \int_S \frac{K \, I(\mathbf{r}'_1) \, e^{-i\bar{k}(|\mathbf{r}_2 - \mathbf{r}'_2| - |\mathbf{r}_1 - \mathbf{r}'_1|)} \delta(\mathbf{r}'_1 - \mathbf{r}'_2)}{\bar{\lambda}^2 |\mathbf{r}_2 - \mathbf{r}'_2| |\mathbf{r}_1 - \mathbf{r}'_1|} \, d^2 r'_1 \, d^2 r'_2 \\ &\approx \frac{K}{\bar{\lambda}^2 R^2} \int_S I(\mathbf{r}'_1) \, e^{-i\bar{k}(|\mathbf{r}_2 - \mathbf{r}'_1| - |\mathbf{r}_1 - \mathbf{r}'_1|)} \, d^2 r'_1 \\ &\approx \frac{K}{\bar{\lambda}^2 R^2} e^{i\psi} \int_S e^{i\bar{k}(\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{r}'_1 / R} \, d^2 r'_1 \end{aligned}$$

◦ This is "just" a diffraction integral!

PROPAGATION OF THE MUTUAL INTENSITY

$$J(\mathbf{r}_1,\mathbf{r}_2) = \int_S\int_S \frac{J(\mathbf{r}'_1,\mathbf{r}'_2)e^{-ik(|\mathbf{r}_2-\mathbf{r}'_2|-|\mathbf{r}_1-\mathbf{r}'_1|)}}{\overline{\lambda}^2|\mathbf{r}_2-\mathbf{r}'_2||\mathbf{r}_1-\mathbf{r}'_1|}\,d^2r'_1\,d^2r'_2$$



COHERENCE THEORY (11)

- Example: Uniformly bright, incoherent source
  - ▷ Source area = A_S
  - ▷ Coherence area at a large distance R:

$$A_{\mathrm{c}} = \frac{(\overline{\lambda} R)^2}{A_S}$$

  - ▷ For a star, $\sqrt{A_{\mathrm{c}}}$ is of the order of a few meters to hundreds of meters (neglecting atmospheric effects)
    - ◦ Starlight *is* special... it's coherent!
    - ◦ Michelson stellar interferometer
    - ◦ Hanbury Brown's intensity interferometer