THE UNIVERSITY OF TEXAS AT DALLAS

Physical Optics

EE 6317

Summer 2008
Assignment 6

Due Date:

July 23, 2008

Reading:

Eugene Hecht, *Optics*, Fourth Edition, Chapters 7 and 8

Reference Material:

Dr. Cantrell's slides on diffraction theory
Dr. Cantrell's notes on diffraction theory

Problems:

Please write your answers to the following problems on engineering paper. No credit will be given for work handed in on other types of paper.

1. In lecture, the function

$$u(\mathbf{r}) = A\,\frac{e^{ikr}}{r} \tag{1}$$

was shown to be an outgoing spherical wave emitted by a point source. Verify that

$$\nabla u(\mathbf{r}) = \hat{\mathbf{r}}\,A\frac{e^{ikr}}{r}\left(ik - \frac{1}{r}\right) . \tag{2}$$

2. Consider the one-dimensional Laplace equation

$$\frac{d^2}{dx^2}f(x) = 0, \tag{3}$$

with the boundary conditions $f(0) = f(1) = 0$. Show that the Green function $G(x, x')$ such that

$$\frac{d^2}{dx^2}G(x, x') = \delta(x - x'), \tag{4}$$

G is continuous at $x = x'$, and

$$G(0, x') = G(1, x') = 0, \tag{5}$$

is

$$G(x, x') = \begin{cases} (x' - 1)x, & \text{if } x < x'; \\ x'(x - 1), & \text{if } x > x'. \end{cases} \tag{6}$$

[Hint: Use a signal-space approach with the following steps: Start by solving the homogeneous equation for $x < x'$ and $x > x'$ and the given boundary conditions. Require that $G(x, x')$ be continuous at $x = x'$. Then integrate both sides of Eq. (4) with respect to x from $x = x' - \epsilon$ to $x = x' + \epsilon$, and take the limit as $\epsilon \to 0$, to get the discontinuity in the slope of G at $x = x'$.]

3. Verify that

$$u(\rho, \phi, z) = J_m(\alpha\rho)\, e^{im\phi}\, e^{i\beta z} \tag{7}$$

where

$$k^2 = \alpha^2 + \beta^2 \tag{8}$$

is a solution of the Helmholtz equation

$$(\nabla^2 + k^2)u = 0. \tag{9}$$

[Hint: it will be useful to know that J_m, the Bessel function of the first kind of order m, satisfies the equation

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \left(1 - \frac{m^2}{z^2}\right)\right] J_m(z) = 0.] \tag{10}$$

4. Show heuristically that

$$\int_{-\infty}^{\infty} e^{\pm ix^2}\, dx = e^{\pm i\pi/4}\sqrt{\pi}. \tag{11}$$

[Hint: Let $x^2 = \pm iy^2$.] Also obtain this result by evaluating the integral

$$\int_{-\infty}^{\infty} e^{-\epsilon x^2 \pm ix^2}\, dx \quad \text{where} \quad \epsilon > 0$$

and taking the limit $\epsilon \to 0$.

5. Show (by completing the square in the exponent) that

$$\int_{-\infty}^{\infty} e^{-i(x^2 + kx)}\, dx = e^{-i\pi/4}\sqrt{\pi} e^{i(k/2)^2}. \tag{12}$$

This is the Fourier transform of the function e^{-ix^2}.

6. This problem pertains to Young's two-slit interference experiment, in which the distance between the slit centers is $d = 1$ mm, the slit-to-screen distance is $D = 1$ m, and the wavelength is $\lambda = 500$ nm. The point of the problem is to take account of diffraction due to the finite width of the slits.

   (a) The irradiance of the fringes depends on both the interference of light from two slits and the diffraction pattern of a single slit. Write down an expression for the irradiance on the observation screen, taking both interference and diffraction into account. No derivation is required.

   (b) Find the spacing between adjacent interference fringes on the screen.

   (c) Find the width of the central maximum of the diffraction pattern, assuming that the slit width is $w = 0.1$ mm.

   (d) Sketch the resulting two-slit diffraction pattern.

   (e) How many maxima of the Young interference pattern will fall under the central diffraction maximum if the slit width is reduced to $w = 0.01$ mm?

7. Inspection of a Google Earth image of downtown Dallas shows that the reflection of the Sun on an automobile's front hood is imaged as a circular spot with a diameter equal to the width of the hood. Assuming that the hood is 2 m wide, the wavelength of light is $\lambda = 5 \times 10^{-7}$ m, the altitude of the satellite that obtained the image is 200 km, and the satellite camera had a circular aperture, find the minimum aperture diameter.

8. This problem concerns Fresnel diffraction of light from a point source by a straight edge. The perpendicular distance from the point source to the straight edge is $r' = 2$ m, the perpendicular distance from the straight edge to the observation screen is $s' = 2$ m, the wavelength is $\lambda = 5 \times 10^{-7}$ m, and a straight line from the point source to the observation point, perpendicular to the plane that contains the straight edge, passes a distance 1.0 mm from the straight edge. The observation point is not in the geometrical shadow of the straight edge. Draw the line that represents the field value at the observation point on the Cornu spiral below.