FIBER PARAXIAL WAVE EQUATION (1)

- Eliminate $\mathbf{H} = \mathbf{B}$ from the Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}$$

by calculating $\nabla \times (\nabla \times \mathbf{E})$ and making the approximation

$$\nabla(\nabla \cdot \mathbf{E}) = -4\pi\nabla(\nabla \cdot \mathbf{P}) = \mathbf{0}$$

(OK for weak nonlinearities, as in telecommunications fibers)

- Wave equation:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} = \frac{4\pi\sigma}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

FIBER PARAXIAL WAVE EQUATION (2)

- Assume that $\mathbf{E}$ is a pulse in the HE_{11} mode:

$$\mathbf{E}(\mathbf{r}_T, z, t) = \mathrm{Re}\left[\hat{\mathbf{e}}\, e^{i(\beta_0 z - \omega_0 t)} \int_{-\infty}^{\infty} \psi_m(\mathbf{r}_T, \omega)\, \tilde{\mathcal{E}}(z, \omega)\, e^{-i(\omega - \omega_0)t}\, d\omega\right]$$

where ψ_m is the mode function for mode m:

$$\left(\nabla_T^2 + \frac{\omega^2}{c^2}\epsilon(\mathbf{r}_T, \omega) - [\beta(\omega)]^2\right)\psi_m(\mathbf{r}_T, \omega) = 0$$

- ▷ Slowly varying field envelope: $\boxed{\mathcal{E}(z, t) = \int_{-\infty}^{\infty} \tilde{\mathcal{E}}(z, \omega)\, e^{-i(\omega - \omega_0)t}\, d\omega}$
- ▷ Attenuation term:

$$\frac{4\pi\sigma}{c^2}\frac{\partial \mathbf{E}}{\partial t} = -\frac{4\pi i\sigma}{c^2}\hat{\mathbf{e}}\, e^{i(\beta_0 z - \omega_0 t)} \int_{-\infty}^{\infty} \omega\, \psi_m(\mathbf{r}_T, \omega)\, \tilde{\mathcal{E}}(z, \omega)\, e^{-i(\omega - \omega_0)t}\, d\omega$$

- ▷ Intensity attenuation coefficient: $\alpha \approx \dfrac{4\pi\omega_0\sigma}{\beta c^2}$

FIBER PARAXIAL WAVE EQUATION (3)

- Include the frequency dependence of the dielectric constant:

$$\mathbf{D}(\mathbf{r}_T, z, t) = \mathrm{Re}\left[\hat{\mathbf{e}}\, e^{i(\beta_0 z - \omega_0 t)} \int_{-\infty}^{\infty} \epsilon(\mathbf{r}_T, \omega)\, \psi_m(\mathbf{r}_T, \omega)\, \tilde{\mathcal{E}}(z, \omega)\, e^{-i(\omega-\omega_0)t}\, d\omega\right] + 4\pi \mathbf{P}_{NL}$$

Then

$$-\frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} = \mathrm{Re}\left[\hat{\mathbf{e}}\, e^{i(\beta_0 z - \omega_0 t)} \int_{-\infty}^{\infty} \frac{\omega^2 \epsilon(\mathbf{r}_T, \omega)}{c^2}\, \psi_m(\mathbf{r}_T, \omega)\, \tilde{\mathcal{E}}(z, \omega)\, e^{-i(\omega-\omega_0)t}\, d\omega\right] - \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}$$

FIBER PARAXIAL WAVE EQUATION (4)

- Separate ∇^2 into transverse and longitudinal parts:

$$\nabla^2 = \nabla_T^2 + \frac{\partial^2}{\partial z^2}$$

$$\nabla_T^2 \mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{e}}\, e^{i(\beta_0 z - \omega_0 t)} \int_{-\infty}^{\infty} [\nabla_T^2 \psi_m(\mathbf{r}_T, \omega)] \tilde{\mathcal{E}}(z,\omega)\, e^{-i(\omega-\omega_0)t}\, d\omega\right]$$

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = \mathrm{Re}\left[\hat{\mathbf{e}}\, e^{i(\beta_0 z - \omega_0 t)} \int_{-\infty}^{\infty} \left(-\beta_0^2 \tilde{\mathcal{E}} + 2i\beta_0 \frac{\partial \tilde{\mathcal{E}}}{\partial z} + \frac{\partial^2 \tilde{\mathcal{E}}}{\partial z^2}\right) \times \psi_m(\mathbf{r}_T, \omega)\, e^{-i(\omega-\omega_0)t}\, d\omega \right]$$

  - ▷ In the **slowly-varying-envelope approximation** (SVEA), neglect $\partial^2 \tilde{\mathcal{E}}/\partial z^2$ in comparison with $\beta_0 \partial \tilde{\mathcal{E}}/\partial z$
  - ▷ The SVEA is not valid for femtosecond pulses

FIBER PARAXIAL WAVE EQUATION (5)

- Equation for the electric-field envelope:

$$\int_{-\infty}^{\infty} \left(2i\beta_0 \frac{\partial \tilde{\mathcal{E}}}{\partial z} + \underbrace{([\beta(\omega)]^2 - \beta_0^2)\tilde{\mathcal{E}}}_{\text{dispersion}} \right) \psi_m(\mathbf{r}_T, \omega)\, e^{-i(\omega-\omega_0)t}\, d\omega$$

$$= -\frac{4\pi i \sigma \omega_0 \mathcal{E}}{c^2}\, \psi_m(\mathbf{r}_T, \omega_0) + \frac{4\pi}{c^2} e^{-i(\beta_0 z - \omega_0 t)} \underbrace{\hat{\mathbf{e}} \cdot \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}}_{\text{nonlinear effects}}$$

  - ▷ Expand β in a Taylor series about $\omega = \omega_0$:

$$\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \tfrac{1}{2}\beta_2(\omega - \omega_0)^2 + \tfrac{1}{6}\beta_3(\omega - \omega_0)^3 + \cdots$$

  - ▷ Information about group-velocity dispersion is contained in the coefficients

$$\beta_m := \left[\frac{d^m \beta}{d\omega^m} \right]_{\omega=\omega_0}$$

FIBER PARAXIAL WAVE EQUATION (6)

- Replace powers of $\omega - \omega_0$ with time derivatives:
  - ▷ Approximate $\Delta\beta^2$:

$$[\beta(\omega)]^2 - \beta_0^2 \approx 2\beta_0 \left[\beta(\omega) - \beta_0\right]$$

  - ▷ Expand the propagation constant:

$$\beta(\omega) - \beta_0 = \beta_1(\omega - \omega_0) + \tfrac{1}{2}\beta_2(\omega - \omega_0)^2 + \tfrac{1}{6}\beta_3(\omega - \omega_0)^3 + \cdots$$

  - ▷ Introduce time derivatives of the field envelope:

$$\begin{aligned}
&\textstyle\int_{-\infty}^{\infty} (\omega-\omega_0)^n \, \tilde{\mathcal{E}}(\mathbf{r}_T, \omega) \, \psi_m(\mathbf{r}_T, \omega) \, e^{-i(\omega-\omega_0)t} \, d\omega \\
&\qquad \approx \psi_m(\mathbf{r}_T, \omega_0) \left(i \frac{\partial}{\partial t} \right)^n \textstyle\int_{-\infty}^{\infty} \tilde{\mathcal{E}}(\mathbf{r}_T, \omega) \; e^{-i(\omega-\omega_0)t} \, d\omega \\
&\qquad \approx i^n \, \psi_m(\mathbf{r}_T, \omega_0) \, \frac{\partial^n \mathcal{E}(z, t)}{\partial t^n}
\end{aligned}$$

FIBER PARAXIAL WAVE EQUATION (7)

- Equation for slowly varying envelope of $\mathbf{E}$:

$$\left[2i\beta_0\frac{\partial}{\partial z} + 2\beta_0\left(i\beta_1\frac{\partial}{\partial t} - \frac{\beta_2}{2}\frac{\partial^2}{\partial t^2} - i\frac{\beta_3}{6}\frac{\partial^3}{\partial t^3}\right)\right]\mathcal{E}(z,t)\,\psi_m(\mathbf{r}_T,\omega_0)$$

$$= -\frac{4\pi i\sigma\omega_0\mathcal{E}}{c^2}\,\psi_m(\mathbf{r}_T,\omega_0) + \frac{4\pi}{c^2}e^{-i(\beta_0 z-\omega_0 t)}\hat{\mathbf{e}}\cdot\frac{\partial^2\mathbf{P}_{NL}}{\partial t^2}$$

  - ▷ Project both sides on $\psi_m(\mathbf{r}_T,\omega_0)$:

$$\left[\frac{\partial}{\partial z} + \left(\beta_1\frac{\partial}{\partial t} + i\frac{\beta_2}{2}\frac{\partial^2}{\partial t^2} - \frac{\beta_3}{6}\frac{\partial^3}{\partial t^3}\right)\right]\mathcal{E}(z,t)$$

$$= -\frac{\alpha}{2}\mathcal{E} - \frac{2\pi i}{\beta_0 c^2}\frac{e^{-i(\beta_0 z-\omega_0 t)}\int_{\mathcal{A}}\psi_m(\mathbf{r}_T,\omega_0)^*\hat{\mathbf{e}}\cdot\frac{\partial^2\mathbf{P}_{NL}}{\partial t^2}d^2\mathbf{r}_T}{\int_{\mathcal{A}}\left|\psi_m(\mathbf{r}_T,\omega_0)\right|^2 d^2\mathbf{r}_T}$$

FIBER PARAXIAL WAVE EQUATION (8)

- If $\beta_2 = \beta_3 = 0$, $\alpha = 0$, and $\mathbf{P}_{NL} = 0$, then

$$\frac{\partial \mathcal{E}}{\partial z} = \beta_1 \frac{\partial \mathcal{E}}{\partial t} \quad \Rightarrow \quad \mathcal{E}(z,t) = \mathcal{E}(0, t - \beta_1 z),$$

which describes the propagation of a pulse with **group velocity**

$$v_g = \frac{1}{\beta_1}.$$

▷ Transform to a frame moving with the pulse:

$$t' = t - \beta_1 z, \quad z' = z$$

▷ Functions transform as follows:

$$\bar{f}(z',\, t') := f(z,\, t)$$

$$\frac{\partial \bar{f}}{\partial z'} = \frac{\partial f}{\partial z} + \beta_1 \frac{\partial f}{\partial t}$$

FIBER PARAXIAL WAVE EQUATION (9)

- General fiber paraxial wave equation:

$$\left[\frac{\partial}{\partial z'} + \left(i\frac{\beta_2}{2}\frac{\partial^2}{\partial t'^2} - \frac{\beta_3}{6}\frac{\partial^3}{\partial t'^3}\right)\right]\overline{\mathcal{E}}(z', t') = -\frac{\alpha}{2}\overline{\mathcal{E}} - \frac{2\pi i}{\beta_0 c^2}\frac{e^{-i(\beta_0 z - \omega_0 t)}\int_{\mathcal{A}}\psi_m(\mathbf{r}_T, \omega_0)^* \hat{\mathbf{e}}\cdot\frac{\partial^2 \overline{\mathbf{P}}_{NL}}{\partial t^2} d^2\mathbf{r}_T}{\int_{\mathcal{A}}\left|\psi_m(\mathbf{r}_T, \omega_0)\right|^2 d^2\mathbf{r}_T}$$

- ▷ Group velocity dispersion effects: β_2, β_3
- ▷ Nonlinear effects: $\overline{\mathbf{P}}_{NL}$
- ▷ Mode volume effects: $\int_{\mathcal{A}}\left|\psi_m(\mathbf{r}_T, \omega_0)\right|^2 d^2\mathbf{r}_T$

FIBER PARAXIAL WAVE EQUATION (10)

- Envelope form for the nonlinear polarization at ω_n:

$$\mathbf{P}_{NL,n} = \mathrm{Re}\left[-i\hat{\mathbf{e}}_n \psi_n(\mathbf{r}_T, \omega_n)\mathcal{P}_n(z,\, t)e^{i(\beta_n z - \omega_n t)}\right]$$

- Coupled-wave fiber paraxial wave equation:

$$\left[\frac{\partial}{\partial z'} + \left(i\frac{\beta_2}{2}\frac{\partial^2}{\partial t'^2} - \frac{\beta_3}{6}\frac{\partial^3}{\partial t'^3}\right)\right]\overline{\mathcal{E}}_n(z', t') = -\frac{\alpha}{2}\overline{\mathcal{E}}_n + \frac{2\pi\mu_{nklm}}{\beta(\omega_n)c^2}\left(\omega_n^2 + 2i\omega_n\frac{\partial}{\partial t}\right)\overline{\mathcal{P}}_n(z', t')$$

FIBER PARAXIAL WAVE EQUATION (11)

- Power-series expansion of $\beta(\omega)$ leads to a PDE in t and z
  - ▷ For improved accuracy and numerical stability, use a numerically approximated $\beta(\omega)$ instead of a power-series expansion
  - ▷ Revised general fiber paraxial wave equation in terms of $t' = t - \beta_0 z/\omega_0$:

$$
\left[\frac{\partial}{\partial z'} - \frac{i}{\beta_0}\left([\beta(\Delta\omega)]^2 - \beta_0^2\right)\right]\tilde{\mathcal{E}}(z', \Delta\omega)
= -\frac{\alpha}{2}\tilde{\mathcal{E}} - \frac{i}{\beta_0 c^2}\int_{-\infty}^{\infty}\int_{\mathcal{A}} \frac{e^{i\Delta\omega[t' + \beta_0 z'/\omega_0]}\psi_m(\mathbf{r}_T, \omega_0)^* \left\langle e^{i\omega_0 t'}\hat{\mathbf{e}} \cdot \dfrac{\partial^2 \overline{\mathbf{P}}_{NL}}{\partial t^2}\right\rangle}{\int_{\mathcal{A}} |\psi_m(\mathbf{r}_T, \omega_0)|^2 \, d^2\mathbf{r}_T} \, d^2\mathbf{r}_T \, dt'
$$