Fiberoptic Mode Functions: A Tutorial

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1 Introduction

Maxwell's equations in free space and in linear, isotropic bulk media are linear and invariant under translations (*i.e.*, shifts) in space and time. Shift invariance in time (t) implies that there exist single-frequency solutions proportional to $e^{-i\omega t}$. If one idealizes an optical fiber as an infinitely long cylinder with rotational symmetry about the z axis, then the permittivity ϵ depends only on the radial coordinate r. Translational invariance in the z direction implies that there exist single-spatial-frequency solutions proportional to $e^{i\beta z}$, where β is the propagation constant in the fiber. Then the fundamental solutions of Maxwell's equations in an ideal, linear fiber are of the form

$$\mathbf{E}\left(r,\theta,z,t\right)=\mathbf{e}\left(r,\theta\right)e^{i(\beta z-\omega t)} \tag{1}$$

and

$$\mathbf{H}\left(r,\theta,z,t\right)=\mathbf{h}\left(r,\theta\right)e^{i(\beta z-\omega t)}. \tag{2}$$

The function $e^{i(\beta z-\omega t)}$ carries an irreducible representation of the translation group, and describes a monochromatic (fixed-frequency) field. The boundary conditions (to be discussed later) imply an infinite discrete spectrum of β's, but in practice an optical fiber used for telecommunications is operated at a frequency at which only one mode can propagate.

2 Relation between longitudinal and transverse field components

It is convenient to work separately with the fields' longitudinal components (parallel to the z axis) and transverse components (perpendicular to the z axis):

$$\mathbf{E}=\left(\mathbf{e}_T\left(r,\theta\right)+\hat{\mathbf{z}}e_z\left(t,\theta\right)\right)e^{i(\beta z-\omega t)} \tag{3}$$

$$\mathbf{H} = \left(\mathbf{h}_T\left(r,\theta\right) + \hat{\mathbf{z}} h_z\left(t,\theta\right)\right) e^{i(\beta z-\omega t)} \tag{4}$$

We shall show that it is possible to obtain useful equations for the transverse components $\mathbf{e}_T$ and $\mathbf{h}_T$ in terms of the longitudinal components e_z and h_z.

In Gaussian units (in which $\epsilon_0 = \frac{1}{4\pi}$ and $\mu_0 = \frac{4\pi}{c}$), and in a nonmagnetic material (in which $\mu_{\mathrm{r}} \approx 1$, as in silica glass), $\mathbf{B} = \mathbf{H}$. Then, for a monochromatic field, Faraday's law reads

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \frac{i\omega}{c}\mathbf{H}. \tag{5}$$

The factor $-i\omega$ comes from assuming a harmonic time dependence (monochromatic frequency spectrum) for the fields $\mathbf{E}$ and $\mathbf{B}$. The angular frequency is

$$\omega = ck_0 = c\frac{2\pi}{\lambda_{vac}} \tag{6}$$

where λ_{vac} is the wavelength of the light in vacuum.

Let us assume that the fiber material is a linear, isotropic dielectric, such that in cylindrical coordinates

$$\mathbf{D}(r,\theta,z,t) = \epsilon(r)\,\mathbf{E}(r,\theta,z,t) \tag{7}$$

where ϵ is a scalar function of r alone. Then

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{8}$$

Still assuming that $\mathbf{E}$ has a harmonic time dependence, one finds that Ampère's law implies that

$$\nabla \times \mathbf{H} = -\frac{i\omega\epsilon}{c}\mathbf{E} \tag{9}$$

for time-harmonic (monochromatic) fields.

From Eqs. (5) and (6) one has

$$\nabla \times \mathbf{E} = ik_0\mathbf{H}, \tag{10}$$

which says that $\mathbf{H}$ is $90°$ out of phase with respect to $\mathbf{E}$. Likewise,

$$\nabla \times \mathbf{H} = -ik_0\epsilon\mathbf{E} \tag{11}$$

which implies that $\mathbf{E}$ is $-90°$ out of phase with respect to $\mathbf{H}$. These equations also imply that β is the same for the magnetic and electric fields.

In order to benefit from the separation of a vector field

$$\mathbf{A} = \mathbf{A}_T + \hat{\mathbf{z}}A_z \tag{12}$$

into longitudinal and transverse components, one must also separate $\nabla \times \mathbf{A}$ into longitudinal and transverse components. To do this in cylindrical coordinates, recall that the gradient of a scalar field is

$$\nabla \Psi = \hat{\mathbf{r}} \frac{\partial \Psi}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial \Psi}{\partial \theta} + \hat{\mathbf{z}} \frac{\partial \Psi}{\partial z}, \tag{13}$$

where the component of $\nabla \Psi$ that is orthogonal to $\hat{\mathbf{z}}$ is defined as the transverse gradient of $\nabla \Psi$,

$$\nabla_T \Psi = \hat{\mathbf{r}} \frac{\partial \Psi}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial \Psi}{\partial \theta}. \tag{14}$$

Also in cylindrical coordinates, the curl of a vector field is

$$\nabla \times \mathbf{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r A_\theta \right) - \frac{\partial A_r}{\partial \theta} \right) \hat{\mathbf{z}}. \tag{15}$$

It is useful to consider a special case before tackling the general expression above.

The curl of a vector field $\mathbf{A}_T$ that lies entirely in the transverse plane, *and that is independent of* z, is parallel to $\hat{\mathbf{z}}$. Let

$$\nabla_T \times \mathbf{A}_T := \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r A_\theta \right) - \frac{\partial A_r}{\partial \theta} \right) \hat{\mathbf{z}}. \tag{16}$$

When $\mathbf{A}_T$ is independent of z, this is the formula for $\nabla \times \mathbf{A}_T$. Because the fields in an optical fiber depend on z, the curl of $\mathbf{A}_T$ is not entirely along $\hat{\mathbf{z}}$.

Often it is useful to consider the electromagnetic field vectors of a quasi-monochromatic beam of light as products of a "carrier" function that oscillates sinusoidally at the mean frequency of the light, and an "envelope" function that varies slowly on the spatial scale of an optical wavelength and the temporal scale of an optical period:

$$\mathbf{A}_T = \mathbf{a}_T \left(r, \theta \right) e^{i(\beta z - \omega t)} \tag{17}$$

In this equation, $\mathbf{a}_T$ is the envelope and $e^{i(\beta z - \omega t)}$ is the carrier.

It follows from Eqs. (15–17) that

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}} \frac{1}{r} \frac{\partial A_z}{\partial \theta} - i\beta \hat{\mathbf{r}} A_\theta + i\beta \hat{\boldsymbol{\theta}} A_r - \hat{\boldsymbol{\theta}} \frac{\partial A_z}{\partial r} + \nabla_T \times \mathbf{A}_T. \tag{18}$$

With the help of the identities

$$\hat{\mathbf{z}} \times \nabla_T \Psi = -\hat{\mathbf{r}} \frac{1}{r} \frac{\partial \Psi}{\partial \theta} + \hat{\boldsymbol{\theta}} \frac{\partial \Psi}{\partial r} \tag{19}$$

and

$$i\beta \hat{\mathbf{z}} \times \mathbf{A}_T = -i\beta \hat{\mathbf{r}} A_\theta + i\beta \hat{\boldsymbol{\theta}} A_r. \tag{20}$$

one can collect the terms of Eq. (18) into the equation

$$\boxed{\nabla\times\mathbf{A}=\hat{\mathbf{z}}\times\left[i\beta\mathbf{A}_T-\nabla_T A_z\right]+\nabla_T\times\mathbf{A}_T} \tag{21}$$

holds in cylindrical coordinates for any vector field with the z-dependence $e^{i\beta z}$.

Let us use this expression to evaluate the curl of $\mathbf{E}$:

$$\nabla\times\mathbf{E}=\overbrace{-\hat{\mathbf{z}}\times\nabla_T E_z+i\beta\hat{\mathbf{z}}\times\mathbf{E}_T}^{\text{transverse}}+\overbrace{\nabla_T\times\mathbf{E}_T}^{\text{longitudinal}}=ik_0\mathbf{H}=ik_0\left(\mathbf{H}_T+\hat{\mathbf{z}}H_z\right). \tag{22}$$

Cancelling common exponential factors, one obtains the expression

$$\nabla\times\mathbf{e}=-\hat{\mathbf{z}}\times\nabla_T e_z+i\beta\hat{\mathbf{z}}\times\mathbf{e}_T+\nabla_T\times\mathbf{e}_T=ik_0\left(\mathbf{h}_T+\hat{\mathbf{z}}h_z\right) \tag{23}$$

for the curl of the envelope $\mathbf{e}$. The longitudinal component of any vector field $\mathbf{A}$ is $\hat{\mathbf{z}}\cdot\mathbf{A}$. Thus Eq. (23) implies that

$$\hat{\mathbf{z}}\cdot(\nabla\times\mathbf{e})=\hat{\mathbf{z}}\cdot(\nabla_T\times\mathbf{e}_T)=ik_0h_z. \tag{24}$$

Solving for h_z, one obtains

$$\boxed{h_z=-\frac{i}{k_0}\hat{\mathbf{z}}\cdot\left(\nabla_T\times\mathbf{e}_T\right).} \tag{25}$$

Calculating the transverse components of Eq. (23), one obtains

$$-\hat{\mathbf{z}}\times\nabla_T e_z+i\beta\left(\hat{\mathbf{z}}\times\mathbf{e}_T\right)=ik_0\mathbf{h}_T \tag{26}$$

from which it follows that

$$\boxed{\mathbf{h}_T=\frac{1}{k_0}\hat{\mathbf{z}}\times\left[\beta\mathbf{e}_T+i\nabla_T e_z\right].} \tag{27}$$

These equations determine the longitudinal and transverse components of $\mathbf{h}$ in terms of the longitudinal and transverse components of $\mathbf{e}$.

One can similarly obtain equations for the longitudinal and transverse components of $\mathbf{e}$ in terms of the longitudinal and transverse components of $\mathbf{h}$. By the right-hand rule, the vector field $\hat{\mathbf{z}}\times\mathbf{h}_T$ is purely transverse. Calculating $\hat{\mathbf{z}}\times\mathbf{h}_T$ from Eq. (27) and using the identity $\hat{\mathbf{z}}\times(\hat{\mathbf{z}}\times\mathbf{A})=(\hat{\mathbf{z}}\cdot\mathbf{A})\hat{\mathbf{z}}-(\hat{\mathbf{z}}\cdot\hat{\mathbf{z}})\mathbf{A}$, one gets

$$\begin{aligned}\hat{\mathbf{z}}\times\mathbf{h}_T&=\hat{\mathbf{z}}\times\left[\frac{1}{k_0}\hat{\mathbf{z}}\times\left(i\nabla_T e_z+\beta\mathbf{e}_T\right)\right]\\&=\frac{1}{k_0}\left[\hat{\mathbf{z}}\left(\hat{\mathbf{z}}\cdot\left(i\nabla_T e_z+\beta\mathbf{e}_T\right)\right)-\left(i\nabla e_z+\beta\mathbf{e}_T\right)\left(\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}\right)\right].\end{aligned} \tag{28}$$

Using the relations $\hat{\mathbf{z}} \cdot \nabla_T e_z = 0$ and $\hat{\mathbf{z}} \cdot \beta \mathbf{e}_T = 0$, one finds that

$$\hat{\mathbf{z}} \times \mathbf{h}_T = -\frac{i}{k_0} \left[\nabla_T e_z - i\beta \mathbf{e}_T \right]. \tag{29}$$

At this point we have enough tools to attack the problem of evaluating the longitudinal and transverse components of Ampère's law,

$$\nabla \times \mathbf{H} = \frac{\epsilon}{c} \frac{\partial \mathbf{E}}{\partial t} = -i\epsilon k_0 \mathbf{E}. \tag{30}$$

From Eq. (22) it follows that

$$\nabla \times \mathbf{H} = -\hat{\mathbf{z}} \times \nabla_T H_z + i\beta \left(\hat{\mathbf{z}} \times \mathbf{H}_T \right) + \nabla_T \times \mathbf{H}_T \tag{31}$$

From Eqs. (30–31) one gets

$$-\hat{\mathbf{z}} \times \nabla_T h_z + i\beta \left(\hat{\mathbf{z}} \times \mathbf{h}_T \right) + \nabla_T \times \mathbf{h}_T = -ik_0 \epsilon \mathbf{e}. \tag{32}$$

Calculating the longitudinal components of this equation (by taking the dot product with $\hat{\mathbf{z}}$) yields

$$\hat{\mathbf{z}} \cdot \left(\nabla \times \mathbf{h} \right) = \hat{\mathbf{z}} \cdot \left(\nabla_T \times \mathbf{h}_T \right) = -ik_0 \epsilon e_z, \tag{33}$$

which implies that

$$\boxed{e_z = \frac{i}{k_0 \epsilon} \hat{\mathbf{z}} \cdot \left(\nabla_T \times \mathbf{h}_T \right).} \tag{34}$$

Equating the transverse components of the right-hand and left-hand sides of Eq. (32) gives

$$-\hat{\mathbf{z}} \times \nabla_T h_z + i\beta \left(\hat{\mathbf{z}} \times \mathbf{h}_T \right) = -ik_0 \mathbf{e}_T, \tag{35}$$

which implies that

$$\boxed{\mathbf{e}_T = -\frac{1}{k_0 \epsilon} \hat{\mathbf{z}} \times \left[\beta \mathbf{h}_T + i \nabla_T h_z \right].} \tag{36}$$

Eqs. (25), (27), (34), and (36) can be simplified further with the help of vector identities.

If one substitutes Eq. (27) into Eq. (36), one gets

$$\mathbf{e}_T = -\frac{1}{k_0 \epsilon} \hat{\mathbf{z}} \times \left(\beta \frac{1}{k_0} \hat{\mathbf{z}} \times \left[\beta \mathbf{e}_T + i \nabla_T e_z \right] + i \nabla_T h_z \right). \tag{37}$$

Using the identity

$$\hat{\mathbf{z}} \times \left(\hat{\mathbf{z}} \times \mathbf{C}_T \right) = \hat{\mathbf{z}} \left(\hat{\mathbf{z}} \cdot \mathbf{C}_T \right) - \mathbf{C}_T \left(\hat{\mathbf{z}} \cdot \hat{\mathbf{z}} \right) = -\mathbf{C}_T, \tag{38}$$

where $\mathbf{C}_T$ is a purely transverse vector field, one gets

$$\begin{aligned}\mathbf{e}_T &= -\frac{\beta}{k_0\epsilon}\left[-\frac{1}{k_0}\left(\beta\mathbf{e}_T + i\nabla_T e_z\right)\right] - \frac{i}{k_0\epsilon}\hat{\mathbf{z}}\times(\nabla_T h_z) \\ &= \frac{\beta^2}{k_0^2\epsilon}\left[\mathbf{e}_T + i\frac{k_0}{\beta}\nabla_T e_z\right] - \frac{i}{k_0\epsilon}\hat{\mathbf{z}}\times(\nabla_T h_z)\,.\end{aligned} \tag{39}$$

Collecting the terms in $\mathbf{e}_T$ results in the equation

$$(k_0^2\epsilon - \beta^2)\,\mathbf{e}_T = i\left[\beta\nabla_T e_z - k_0\hat{\mathbf{z}}\times(\nabla_T h_z)\right]. \tag{40}$$

There are two possibilities:

- Both e_z and h_z vanish. In this case either both fields vanish ($\mathbf{e}_T = \mathbf{0} \Rightarrow \mathbf{e} = \mathbf{0} \Rightarrow \mathbf{h} = \mathbf{0}$, or the equation

$$\beta^2 = k_0^2\epsilon \tag{41}$$

  holds. In the latter case, one has a **TEM (transverse electromagnetic) wave**, because both $\mathbf{E}$ and $\mathbf{H}$ are purely transverse. An example is a plane wave in a homogeneous bulk medium. However, Eq. (41) does not hold for guided waves.

- Either e_z or h_z is nonzero, or both are nonzero. In this case $\beta^2 \neq k_0^2\epsilon$, and one can divide both sides of Eq. (40) by $(k_0^2\epsilon - \beta^2)$ to obtain the equation

$$\boxed{\mathbf{e}_T = \frac{i}{k_0^2\epsilon - \beta^2}\left[\beta\nabla_T e_z - k_0\hat{\mathbf{z}}\times(\nabla_T h_z)\right]} \tag{42}$$

  for the transverse component of the electric field in terms of the longitudinal components of both fields.

From Eq. (42) one finds that the radial and azimuthal components of $\mathbf{e}$ are

$$\boxed{e_r = \frac{i}{k_0^2\epsilon - \beta^2}\left[\beta\frac{\partial e_z}{\partial r} + \frac{k_0}{r}\frac{\partial h_z}{\partial \theta}\right]} \tag{43}$$

and

$$\boxed{e_\theta = \frac{i}{k_0^2\epsilon - \beta^2}\left[\beta\frac{\partial e_z}{\partial \theta} - k_0\frac{\partial h_z}{\partial r}\right]}. \tag{44}$$

Similarly,

$$\boxed{\mathbf{h}_T = \frac{i}{k_0^2\epsilon - \beta^2}\left[\beta\nabla_T h_z + k_0\epsilon\hat{\mathbf{z}}\times(\nabla_T e_z)\right]} \tag{45}$$

from which one obtains

$$\boxed{h_r = \frac{i}{k_0^2\epsilon - \beta^2}\left[\beta\frac{\partial h_z}{\partial r} - \frac{k_0\epsilon}{r}\frac{\partial e_z}{\partial \theta}\right]} \tag{46}$$

and

$$\boxed{h_\theta = \frac{i}{k_0^2\epsilon - \beta^2}\left[\frac{\beta}{r}\frac{\partial h_z}{\partial \theta} + k_0\epsilon\frac{\partial e_z}{\partial r}.\right]} \tag{47}$$

These equations determine $\mathbf{e}_T$ and $\mathbf{h}_T$ in terms of e_z and h_z.

3 Wave equations for the longitudinal field components

The next step is to find wave equations for e_z and h_z.

From the two remaining equations of Maxwell one has

$$\nabla \cdot \mathbf{H} = 0 \tag{48}$$

(magnetic monopoles don't exist) and

$$\nabla \cdot \big(\epsilon\mathbf{E}\big) = 0 \tag{49}$$

(no free charges exist in an ideal insulating medium). It follows that $\nabla \cdot \mathbf{E} = 0$ almost everywhere if ϵ is piecewise constant. However, if the dielectric permittivity ϵ is not piecewise constant, then one must expand $\nabla \cdot \big(\epsilon\mathbf{E}\big)$ using the vector identity

$$\nabla \cdot \big(\epsilon\mathbf{E}\big) = \epsilon\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla\epsilon. \tag{50}$$

Transposition yields the expression

$$\nabla \cdot \mathbf{E} = -\frac{1}{\epsilon}\mathbf{E} \cdot \nabla\epsilon, \tag{51}$$

which one can use to eliminate the term $\nabla\big(\nabla \cdot \mathbf{E}\big)$ that arises in the expansion of $\nabla \times \big(\nabla \times \mathbf{E}\big)$ (see below).

In Cartesian coordinates one has

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= (ik_0)\,(-ik_0\epsilon)\,\mathbf{E} \\
&= k_0^2\epsilon\mathbf{E} \\
&= \nabla\,(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} \\
&= -\nabla\left[\frac{1}{\epsilon}\mathbf{E} \cdot \nabla\big(\epsilon\big)\right] - \nabla^2\mathbf{E} \\
&= -\nabla\Big[\mathbf{E} \cdot \nabla\big(\ln\epsilon\big)\Big] - \nabla^2\mathbf{E}.
\end{aligned} \tag{52}$$

In a weakly guiding fiber, in which the total refractive index variation is of the order of 10^{-3}, $\nabla\big(\ln\epsilon\big)$ is usually neglected. Then

$$\nabla\times(\nabla\times\mathbf{E}) = -\nabla^2\mathbf{E} \tag{53}$$

in Cartesian coordinates, and one obtains a Helmholtz equation for $\mathbf{E}$,

$$\left(\nabla^2 + k_0^2\epsilon\right)\mathbf{E} = 0, \tag{54}$$

which implies a Helmholtz equation for the z component,

$$\left(\nabla^2 + k_0^2\epsilon\right)E_z = 0. \tag{55}$$

This equation holds also in cylindrical coordinates, where

$$\nabla^2 = \nabla_T^2 + \frac{\partial^2}{\partial z^2} \tag{56}$$

and

$$\nabla_T^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{57}$$

Therefore

$$\begin{aligned}\nabla^2 E_z &= \nabla_T^2 E_z + \overbrace{\frac{\partial^2}{\partial z^2}\left(e_z\left(r,\theta\right)e^{i(\beta z-\omega t)}\right)}^{-\beta^2 e_z e^{i(\beta z-\omega t)}} \\ &= \left[\left(\nabla_T^2 - \beta^2\right)e_z\right]e^{i(\beta z-\omega t)}, \\ &= -k_0^2\epsilon E_z\end{aligned} \tag{58}$$

which implies that the longitudinal components of $\mathbf{E}$ and $\mathbf{H}$ satisfy the Helmholtz equation:

$$\boxed{\left(\nabla_T^2 + k_0^2\epsilon - \beta^2\right)e_z = 0} \tag{59}$$

and

$$\boxed{\left(\nabla_T^2 + k_0^2\epsilon - \beta^2\right)h_z = 0} \tag{60}$$

For $k_0^2\epsilon-\beta^2 > 0$ these equations predict propagating waves, and for $k_0^2\epsilon-\beta^2 < 0$ they predict exponentially decaying or growing fields.

4 Modes in a step-index fiber

The Helmholtz equations for e_z and h_z derived in the previous section are completely general, but one must supplement them with boundary conditions in

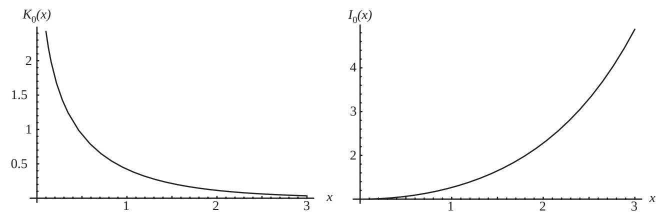


Figure 1: Bessel functions of imaginary argument.

order to obtain a full solution. To show how to do this, we solve the e_z equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial e_z}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 e_z}{\partial \theta^2} + \left(k_0^2\epsilon - \beta^2\right) e_z = 0, \tag{61}$$

for the special case of a step-index fiber. Assume a solution of the separated form

$$e_z = f_m\left(r\right) e^{im\theta}. \tag{62}$$

Then

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df_m}{dr}\right) + \frac{m^2}{r^2} f_m + \left(k_0^2\epsilon - \beta^2\right) f_m = 0. \tag{63}$$

This is Bessel's equation. Therefore the solution is a linear combination of Bessel functions of order m:

$$f_m(r) = \begin{cases} AJ_m\left(\left[k_0^2\epsilon - \beta^2\right]^{\frac{1}{2}} r\right) + BY_m\left(\left[k_0^2\epsilon - \beta^2\right]^{\frac{1}{2}} r\right) & \text{if } k_0^2\epsilon - \beta^2 > 0 \\ CK_m\left(\left[\beta^2 - k_0^2\epsilon\right]^{\frac{1}{2}} r\right) + DI_m\left(\left[\beta^2 - k_0^2\epsilon\right]^{\frac{1}{2}} r\right) & \text{if } k_0^2\epsilon - \beta^2 < 0 \end{cases} \tag{64}$$

J_m is finite at the origin, but Y_m is singular there. I_m grows exponentially for large arguments, while K_m decays exponentially (see Figure 1).

In order to have propagating waves for $r < a$, one must have $k_0^2 n_1^2 - \beta^2 > 0$. The requirement that e_z be finite at $r = 0$ implies that $B = 0$, *i.e.*, that one only has J_m in the fiber core. The requirement that $e_z \to 0$ as $r \to \infty$ implies that if $k_0^2 n_2^2 - \beta^2 < 0$ then $D = 0$.

If $k_0 n_2 < \beta < k_0 n_1$, then there are radially oscillating waves in the core and damped waves in the cladding. Such a mode is called "guided". On the other hand, if $k_0 n_2 < k_0 n_1 < \beta$, then there are damped waves everywhere. Such a mode is called "leaky". If one has $\beta < k_0 n_2 < k_0 n_1$, then there are radially oscillating waves in both the core and the cladding. Such modes are called "radiation modes".

In order to compress the notation, let the radial propagation constant in the core be

$$\boxed{\kappa = \left[k_0^2 n_1^2 - \beta^2\right]^{\frac{1}{2}}} \tag{65}$$

and let the damping rate of the evanescent waves in the cladding be

$$\boxed{\gamma = \left[\beta^2 - k_0^2 n_2^2\right]^{\frac{1}{2}}.} \tag{66}$$

It follows that κ and γ obey the identity

$$\kappa^2 + \gamma^2 = k_0^2 \left(n_1^2 - n_2^2\right). \tag{67}$$

In this equation, $k_0 = 2\pi/\lambda_0$, and $n_1^2 - n_2^2$ is equal to the square of the numerical aperture of the fiber.

For a guided mode and in the fiber core ($r < a$), the longitudinal components are

$$e_z(r,\theta) = \sum_{m=0}^{\infty} c_m^{(1)} J_m(\kappa r)\, e^{im\theta}, \tag{68}$$

and

$$h_z(r,\theta) = \sum_{m=0}^{\infty} d_m^{(1)} J_m(\kappa r)\, e^{im\theta}. \tag{69}$$

In the cladding ($r > a$),

$$e_z(r,\theta) = \sum_{m=0}^{\infty} c_m^{(2)} K_m(\gamma r)\, e^{im\theta} \tag{70}$$

and

$$h_z(r,\theta) = \sum_{m=0}^{\infty} d_m^{(2)} K_m(\gamma r)\, e^{im\theta} \tag{71}$$

In all of these equations the sums run over the guided modes.

4.1 Boundary conditions at the core-cladding interface

At $r = a$, Maxwell's equations require the continuity of the tangential components of $\mathbf{E}, \mathbf{H}$ and the normal components of $\mathbf{D}, \mathbf{B}$. The requirement of continuity of the tangential component of $\mathbf{E}$ reads

$$\lim_{\epsilon \downarrow 0} E_z(a-\epsilon,\theta,z,t) = \lim_{\epsilon \downarrow 0} E_z(a+\epsilon,\theta z,t). \tag{72}$$

Since

$$E_z(r,\theta,z,t) = e_z(r,\theta)\, e^{i(\beta z - \omega t)} \tag{73}$$

one can rewrite Eq. (72) as

$$\lim_{\epsilon \downarrow 0} e_z(a-\epsilon,\theta)\, e^{i(\beta_1 z - \omega t)} = \lim_{\epsilon \downarrow 0} e_z(a+\epsilon,\theta)\, e^{i(\beta_2 z - \omega t)} \tag{74}$$

The z dependence implies that $\beta_1 = \beta_2 = \beta$. Then

$$\lim_{\epsilon \downarrow 0} e_z\,(a - \epsilon, \theta) = \lim_{\epsilon \downarrow 0} e_z\,(a + \epsilon, \theta) \tag{75}$$

This, in turn, implies that

$$\sum_m c_m^{(1)} J_m(\kappa a) e^{im\theta} = \sum_m c_m^{(2)} K_m(\gamma a) e^{im\theta}. \tag{76}$$

Since the Fourier series of a continuous function is unique, one can equate the coefficients of $e^{im\theta}$ on both sides, obtaining

$$c_m^{(1)} J_m(\kappa a) = c_m^{(2)} K_m(\gamma a) \tag{77}$$

as the condition for the continuity of e_z at $r = a$. If β is known, then this equation implies that the ratio $c_m^{(1)}/c_m^{(2)}$ is determined. Following the same steps for the coefficients $d_m^{(1)}$, one finds that the continuity of h_z at $r = a$ implies that

$$d_m^{(1)} J_m(\kappa a) = d_m^{(2)} K_m(\gamma a). \tag{78}$$

We shall show below that, except at the cutoff frequency of the TE and TM modes, $J_m(\kappa a) \neq 0$.

In the limiting case in which

$$\gamma = 0, \tag{79}$$

the function K_m is singular. It follows that when $\gamma = 0$, the coefficients $c_m^{(2)}$ and $d_m^{(2)}$ must vanish in order to keep the right-hand sides of Eqs. (77–78) finite. This implies that the electric and magnetic fields vanish in the core. Since the right-hand sides of Eqs. (77–78) vanish, the left-hand sides vanish also:

$$\gamma = 0 \;\Rightarrow\; c_m^{(1)} J_m(\kappa a) = 0 \text{ and } d_m^{(1)} J_m(\kappa a) = 0. \tag{80}$$

Except in the trivial case in which the fields vanish everywhere, $c_m^{(1)} \neq 0$ and $d_m^{(1)} \neq 0$. Therefore

$$\gamma = 0 \;\Rightarrow\; J_m(\kappa a) = 0. \tag{81}$$

We shall see soon that this condition (for $m = 0$) defines the cutoff frequency below which only a single mode can propagate in the fiber.

Because normal $\mathbf{B}$ is continuous at $r = a$ and $\mathbf{H} = \mathbf{B}$, it follows that the normal (radial) component of $\mathbf{H}$ is continuous at $r = a$. Then

$$\lim_{\epsilon \downarrow 0} h_r\,(a - \epsilon, \theta) = \lim_{\epsilon \downarrow 0} h_r\,(a + \epsilon, \theta) \tag{82}$$

One has

$$h_r\,(r, \theta) = \frac{i}{k_0^2 \epsilon - \beta^2} \left[\beta \frac{\partial h_z}{\partial r} - \frac{k_0 \epsilon}{r} \frac{\partial e_z}{\partial \theta} \right] \tag{83}$$

Therefore

$$\lim_{\epsilon \downarrow 0} \frac{i}{\kappa^2} \left[\beta \frac{\partial h_z}{\partial r} (a-\epsilon, \theta) - \frac{k_0 n_1^2}{a} \frac{\partial e_z}{\partial \theta} (a-\epsilon, \theta) \right] = \lim_{\epsilon \downarrow 0} \frac{i}{-\gamma^2} \left[\beta \frac{\partial h_z}{\partial r} (a+\epsilon, \theta) - \frac{k_0 n_2^2}{a} \frac{\partial e_z}{\partial \theta} (a+\epsilon, \theta) \right]. \tag{84}$$

The continuity of normal **D**, where **D** $= \epsilon$**E**, implies that

$$\lim_{\epsilon \downarrow 0} n_1^2 e_r (a-\epsilon, \theta) = \lim_{\epsilon \downarrow 0} n_2^2 e_r (a+\epsilon, \theta) \tag{85}$$

Then

$$\lim_{\epsilon \downarrow 0} \frac{i n_1^2}{\kappa^2} \left[\beta \frac{\partial e_z}{\partial r} (a-\epsilon, \theta) - \frac{k_0}{a} \frac{\partial h_z}{\partial \theta} (a-\epsilon, \theta) \right] = \lim_{\epsilon \downarrow 0} \frac{i n_2^2}{-\gamma^2} \left[\beta \frac{\partial e_z}{\partial r} (a+\epsilon, \theta) - \frac{k_0}{a} \frac{\partial h_z}{\partial \theta} (a+\epsilon, \theta) \right]. \tag{86}$$

For a step-index fiber, then, the continuity of the normal component of **B** implies that

$$\frac{i}{\kappa^2} \left[\beta d_m^{(1)} \kappa J_m'(\kappa a) - \frac{k_0 n_1^2}{a} c_m^{(1)} i m J_m(\kappa a) \right] = \frac{i}{-\gamma^2} \left[\beta d_m^{(2)} \gamma K_m'(\gamma a) - \frac{k_0 n_2^2}{a} c_m^{(2)} i m K_m(\gamma a) \right] \tag{87}$$

and the continuity of the normal component of **D** implies that

$$\frac{i n_1^2}{\kappa^2} \left[\beta c_m^{(1)} \kappa J_m'(\kappa a) + \frac{k_0}{a} d_m^{(1)} i m J_m(\kappa a) \right] = \frac{i n_2^2}{-\gamma^2} \left[\beta c_m^{(2)} \gamma K_m'(\gamma a) + \frac{k_0}{a} d_m^{(2)} i m K_m(\gamma a) \right]. \tag{88}$$

Again for a step-index fiber, the continuity of the tangential components of **E** and **H** implies that

$$c_m^{(2)} = \frac{J_m(\kappa a)}{K_m(\kappa a)} c_m^{(1)} \tag{89}$$

and

$$d_m^{(2)} = \frac{J_m(\kappa a)}{K_m(\kappa a)} d_m^{(1)}. \tag{90}$$

4.2 Transverse electric modes

Before considering the general case, in which $m \neq 0$, we study the special case of a mode with no azimuthal dependence, that is, a mode such that $e^{im\theta} = 1$

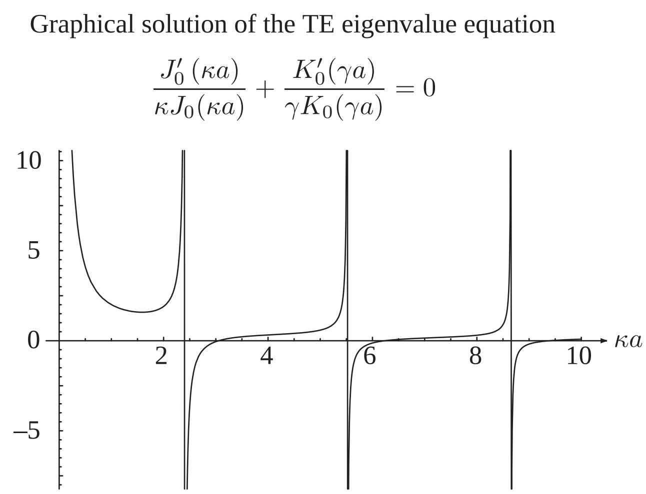


Figure 2: Graphical solution of the eigenvalue equation for $TE_{0\mu}$ modes. In this figure, γ has been set equal to κ.

and therefore $m = 0$. In this special case, the continuity of B_r implies that

$$\begin{aligned} d_0^{(1)} \frac{1}{\kappa^2} \beta\kappa J_0'(\kappa a) &= -\frac{1}{\gamma^2}\beta\gamma K_0'(\gamma a) d_0^{(2)} \\ &= -\frac{1}{\gamma^2}\beta\gamma \frac{K_0'(\gamma a)}{K_0(\gamma a)} J_0(\kappa a) d_0^{(1)} \end{aligned} \tag{91}$$

Collecting terms, one obtains the condition

$$d_0^{(1)} \left[\frac{J_0'(\kappa a)}{\kappa J_0(\kappa a)} + \frac{K_0'(\gamma a)}{\gamma K_0(\gamma a)} \right] J_0(\kappa a) = 0. \tag{92}$$

We shall see soon that $J_0(\kappa a) = 0$ only at cutoff. Then either $d_0^{(1)}$ or the quantity in square brackets vanishes. Assume for now that $h_z \neq 0$. Therefore $d_0^{(1)} \neq 0$ and

$$\boxed{\frac{J_0'(\kappa a)}{\kappa J_0(\kappa a)} + \frac{K_0'(\gamma a)}{\gamma K_0(\gamma a)} = 0} \tag{93}$$

This eigenvalue equation for β has a countably infinite set of roots, which we index with a positive integer μ. Fig. 2 illustrates a graphical solution of the eigenvalue equation.

We shall show that Eq. (93) determines the propagation constant β of a **transverse electric** mode, in which $e_z = 0$. One gives a transverse electric mode the label $TE_{0\mu}$.

To understand the physical significance of Eq. (93), we shall show, using a sequence of logically reversible steps, that when $m = 0$ the equation

$$e_z = 0 \tag{94}$$

implies Eq. (93). It will follow then that Eq. (93) implies that $e_z = 0$, that is, that the mode is transverse electric.

If $m = 0$, then h_z, h_r, h_θ, e_r and e_θ are independent of θ. Eqs. (46) and (47) imply that when $e_z = 0$ and h_z is independent of θ,

$$h_\theta = 0 \tag{95}$$

and

$$h_r = \begin{cases} \dfrac{i\beta}{\kappa^2}\dfrac{\partial h_z}{\partial r}, & \text{if } r < a; \\ \dfrac{-i\beta}{\gamma^2}\dfrac{\partial h_z}{\partial r}, & \text{if } r > a. \end{cases} \tag{96}$$

Thus the magnetic field of a transverse electric mode has only a radial and a longitudinal component. For $e_z = 0$ and no θ dependence, Eqs. (43) and (44) imply that

$$e_r = 0 \tag{97}$$

and

$$e_\theta = \begin{cases} \dfrac{-ik_0}{\kappa^2}\dfrac{\partial h_z}{\partial r}, & \text{if } r < a; \\ \dfrac{ik_0}{\gamma^2}\dfrac{\partial h_z}{\partial r}, & \text{if } r > a. \end{cases} \tag{98}$$

In other words, a transverse electric field has only an azimuthal component.

For a transverse electric mode, the continuity of the normal component of $\mathbf{H}$ implies that h_r is continuous at $r = a$. The continuity of the tangential component of $\mathbf{E}$ implies that at $r = a$,

$$\lim_{\epsilon \downarrow 0} \left[\frac{-ik_0}{\kappa^2}\frac{\partial h_z}{\partial r}(r-\epsilon) - \frac{ik_0}{\gamma^2}\frac{\partial h_z}{\partial r}(r+\epsilon) \right] = 0. \tag{99}$$

It is easy to see that this equation implies Eq. (91), which, in turn, implies Eq. (92). Since $d_0^{(1)} \neq 0$, Eq. (93) follows.

Summary: When $m = 0$, $e_z = 0$ implies Eq. (93) and conversely.

4.3 Transverse magnetic modes

Again for $m = 0$, the continuity of the radial component D_r implies that

$$c_0^{(1)} \left[n_1^2 \frac{J_0'(\kappa a)}{\kappa J_0(\kappa a)} + n_2^2 \frac{K_0'(\gamma a)}{\gamma K_0(\gamma a)} \right] J_0(\kappa a) = 0 \tag{100}$$

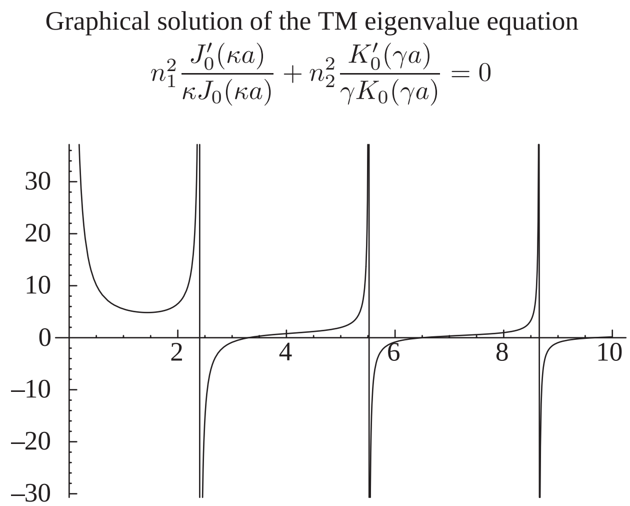


Figure 3: Graphical solution of the eigenvalue equation for $TM_{0\mu}$ modes. In this figure, γ has been set equal to κ. The refractive indices $n_1 = 2.00$ and $n_2 = 1.50$ have been used in order to make this graph distinguishable from the graph for TE modes.

If the longitudinal component of **E** is nonzero, then $e_z \neq 0$ and therefore $c_0^{(1)} \neq 0$. One obtains the eigenvalue equation

$$\boxed{n_1^2 \frac{J_0'(\kappa a)}{\kappa J_0(\kappa a)} + n_2^2 \frac{K_0'(\gamma a)}{\gamma K_0(\gamma a)} = 0} \tag{101}$$

for the propagation constant β. It is straightforward to show that this equation is equivalent to the statement that

$$h_z = 0, \tag{102}$$

which defines a **transverse magnetic** mode. A different TM mode corresponds to each of the infinitely many solutions of the eigenvalue equation (101). The μ^{th} solution defines the $TM_{0\mu}$ mode.

A TE and a TM mode cannot have the same frequency, for if they have the same frequency then the same propagation constant β satisfies both the TE and the TM eigenvalue equations. Then the TE mode is also TM. However, Eqs. (42) and (45) imply that if $e_z = 0$ and $h_z = 0$, then all of the components of both fields vanish.

4.4 The cutoff condition for transverse modes

At cutoff the radial decay constant is zero in the cladding,

$$\gamma = 0. \tag{103}$$

To propagate below the cutoff frequency would require infinite energy. From the cutoff condition $\gamma = 0$ one obtains

$$\kappa^2 = \kappa^2 + \gamma^2 = k_0^2(n_1^2 - n_2^2) \tag{104}$$

where κ is the radial oscillation constant.

The eigenvalue equation for the propagation constant β when $m = 0$ is

$$\frac{J_0'(\kappa a)}{\kappa J_0(\kappa a)} = \begin{cases} -\dfrac{K_0'(\gamma a)}{\gamma K_0(\gamma a)}, & \text{for a } TE_{0\mu} \text{ mode;} \\ -\dfrac{n_2^2}{n_1^2}\dfrac{K_0'(\gamma a)}{\gamma K_0(\gamma a)}, & \text{for a } TM_{0\mu} \text{ mode.} \end{cases} \tag{105}$$

The Bessel function recurrence relation

$$K_m'(z) = K_{m+1}(z) + \frac{m}{z}K_m(z) \tag{106}$$

implies that

$$K_0'(z) = K_1(z). \tag{107}$$

Then

$$\frac{K_0'(\gamma a)}{\gamma K_0(\gamma a)} = \frac{K_1(\gamma a)}{\gamma K_0(\gamma a)} \underbrace{\longrightarrow}_{\gamma \to 0} 0 \tag{108}$$

We have already shown that when $\gamma = 0$, one has

$$J_0(\kappa a) = 0. \tag{109}$$

Then either this condition or the condition

$$\frac{J_0'(\kappa a)}{\kappa J_0(\kappa a)} = 0 \tag{110}$$

defines the value of $k_0 = \omega/c$ at cutoff. Because cutoff occurs at the lowest frequency, we must find out which of the two preceding equations defines the cutoff frequency by looking at the numerical values:

$$J_0'(x) = 0 \text{ at } x = 0, x \approx 3.83, \ldots,$$
$$J_0(x) = 0 \text{ at } x = 0, x \approx 2.405, \ldots$$

Therefore

$$\boxed{\kappa a \approx 2.405} \tag{111}$$

defines the cutoff frequency. The cutoff condition can also be written as

$$\kappa a = j_{0,1} \tag{112}$$

where $j_{\mu,\nu}$ represents the νth zero of J_μ.

Eqs. (65) and (112) imply that the value of the propagation constant β at cutoff is given by the equation

$${n_1}^2{k_0}^2 - {\beta_c}^2 = (j_{01})^2. \tag{113}$$

Therefore

$$\beta_c^2 = n_1^2k_0^2 - (j_{0,1})^2. \tag{114}$$

The quantity

$$\boxed{V := k_0 a\big(n_1^2 - n_2^2\big)^{\frac{1}{2}}} \tag{115}$$

is called the **waveguide parameter** or the **normalized frequency**. Since at cutoff

$$\kappa^2 = k_0^2\left(n_1^2 - n_2^2\right), \tag{116}$$

it follows that

$$\kappa a = V = k_0 a\big(n_1^2 - n_2^2\big)^{\frac{1}{2}}, \tag{117}$$

which states that at cutoff

$$V = j_{0,1}. \tag{118}$$

The point in Fig. 2 that corresponds to Eq. (111) is the intersection of the y-axis with the leftmost vertical segment of the curve.

4.5 Fields at frequencies below the TE–TM cutoff

Eqs. (94–98) imply that for a TE mode, the only nonzero transverse field components are e_θ and h_r, and that they are both proportional to $J_0'(\kappa r)$. The Poynting vector therfore is directed along $\hat{\mathbf{z}}$, and its magnitude is proportional to $[J_0'(\kappa r)]^2$. However, $J_0'(\kappa r) = 0$ at $r = 0$. Therefore a TE mode is a "doughnut" mode, and cannot have the lowest frequency allowed for a propagating mode. Therefore $m \neq 0$ in the lowest mode.

We return to the boundary conditions, assuming that $m \neq 0$. The requirement of continuity of B_r implies that

$$\begin{aligned}\frac{1}{\kappa^2}&\left[\beta\kappa J_m'(\kappa a)d_m^{(1)} - \frac{k_0{n_1}^2}{a}imJ_m(\kappa a)c_m^{(1)}\right]\\ &= -\frac{1}{\gamma^2}\left[\beta\gamma\frac{J_m(\kappa a)}{K_m(\gamma a)}K_m'(\gamma a)d_m^{(1)} - \frac{k_0n_2^2}{a}\frac{J_m(\kappa a)}{K_m(\gamma a)}imK_m(\gamma a)c_m^{(1)}\right].\end{aligned} \tag{119}$$

Divide by J_m and collect terms:

$$\beta\left[\frac{J'_m(\kappa a)}{\kappa J_m(\kappa a)}+\frac{K'_m(\gamma a)}{\gamma K_m(\gamma a)}\right]d_m^{(1)}-im\frac{k_0}{a}\left(\frac{n_1^2}{\kappa^2}+\frac{n_2^2}{\gamma^2}\right)c_m^{(1)}=0. \tag{120}$$

From the requirement of continuity of D_r one obtains the condition

$$im\frac{k_0}{a}\left[\frac{n_1^2}{\kappa^2}+\frac{n_2^2}{\gamma^2}\right]d_m^{(1)}+\beta\left[n_1^2\frac{J'_m(\kappa a)}{\kappa J_m(\kappa a)}+n_2^2\frac{K'_m(\gamma a)}{\gamma K_m(\gamma a)}\right]c_m^{(1)}=0. \tag{121}$$

These two simultaneous conditions are homogeneous linear equations in the unknowns $c_m^{(1)}$, $d_m^{(1)}$. The condition for a nontrivial solution is that the determinant of the coefficients must vanish:

$$\begin{vmatrix} \beta\left[\frac{J'_m(\kappa a)}{\kappa J_m(\kappa a)}+\frac{K'_m(\gamma a)}{\gamma K_m(\gamma a)}\right] & -im\frac{k_0}{a}\left(\frac{n_1^2}{\kappa^2}+\frac{n_2^2}{\gamma^2}\right) \\ im\frac{k_0}{a}\left(\frac{n_1^2}{\kappa^2}+\frac{n_2^2}{\gamma^2}\right) & \beta\left[n_1^2\frac{J'_m(\kappa a)}{\kappa J_m(\kappa a)}+n_2^2\frac{K'_m(\gamma a)}{\gamma K_m(\gamma a)}\right] \end{vmatrix}=0. \tag{122}$$

When one expands this determinant, one obtains the eigenvalue equation

$$\beta^2\left[\frac{J'_m(\kappa a)}{\kappa J_m(\kappa a)}+\frac{K'_m(\gamma a)}{\gamma K_m(\gamma a)}\right]\left[n_1^2\frac{J'_m(\kappa a)}{\kappa J_m(\kappa a)}+n_2^2\frac{K'_m(\gamma a)}{\gamma K_m(\gamma a)}\right]=m^2\frac{k_0^2}{a^2}\left(\frac{n_1^2}{\kappa^2}+\frac{n_2^2}{\gamma^2}\right)^2. \tag{123}$$

This eigenvalue equation determines β for the "hybrid" modes. The right-hand side can be simplified somewhat:

$$\begin{aligned}\left(\frac{n_1^2}{\kappa^2}+\frac{n_2^2}{\gamma^2}\right) &= \frac{N_1^2\gamma^2+n_2^2\kappa^2}{\kappa^2\gamma^2} \\ &= \frac{n_1^2\left(\beta^2-k_0^2n_2^2\right)+n_2^2\left(k_0^2n_1^2-\beta^2\right)}{\kappa^2\gamma^2} \\ &= \frac{\beta^2\left(n_1^2-n_2^2\right)}{\kappa^2\gamma^2}.\end{aligned} \tag{124}$$

In terms of the normalized frequency, Eq. (115), and the quantities

$$U=\kappa a \tag{125}$$

and

$$W=\gamma a \tag{126}$$

the eigenvalue equation becomes

$$\boxed{\left[\frac{J'_m(U)}{UJ_m(U)}+\frac{K'_m(W)}{WK_m(W)}\right]\left[n_1^2\frac{J'_m(U)}{UJ_m(U)}+n_2^2\frac{K'_m(W)}{WK_m(W)}\right]=\left(\frac{m\beta V^2}{k_0U^2W^2}\right)^2} \tag{127}$$

Graphical solution of the fiber eigenvalue equation

$$\left(\frac{J'_m(U)}{UJ_m(U)}+\frac{K'_m(W)}{WK_m(W)}\right)\left(n_1^2\frac{J'_m(U)}{UJ_m(U)}+n_2^2\frac{K'_m(W)}{WK_m(W)}\right)=\left(\frac{m\beta V^2}{k_0U^2W^2}\right)^2$$

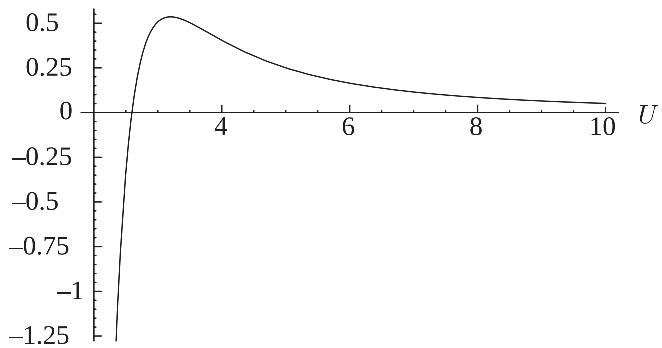


Figure 4: Graphical solution of the full eigenvalue equation. For the purposes of this figure, W has been set equal to U. Other parameters used are $n_1 = 2.00$, $n_2 = 1.50$, $V = 5.5$ and $m = 1$.

This equation determines β when k_0 is given. (Recall that κ and γ, and therefore U and W, depend on β; see Eqs. (65) and (66).)

Because the right-hand side of this equation is positive, it follows that the factors on the left-hand side are either both positive or both negative. If both factors are positive then the resulting mode is labeled $EH_{m\mu}$, where μ is the number of the eigenvalue for a given m, starting at $\mu = 1$. In this case H_z is large. If both factors are negative, then the resulting mode is labeled $HE_{m\mu}$. In this case E_z is large.

Conventionally one plots the **normalized propagation constant**

$$\boxed{b := \frac{\dfrac{\beta}{n_1k_0}-\dfrac{n_2}{n_1}}{1-\dfrac{n_2}{n_1}}} \tag{128}$$

versus the normalized frequency $V = k_0a\left(n_1^2 - n_2^2\right)^{1/2}$. Note that b is defined in such a way that $b = 1$ for a wave propagating in free space:

$$k_0b = \frac{\beta - n_2k_0}{n_1 - n_2} \approx \frac{[\beta(n_2) - n_2k_0] + (n_1 - n_2)\beta'(n_2)}{n_1 - n_2}. \tag{129}$$

As $n_2 \to n_1$, the right-hand side approaches k_0, and therefore b approaches 1. Also, for a wave propagating in a bulk medium with the same refractive index

($n = n_1$) as the core (or for a wave propagating entirely in the core), $\beta = n_1 k_0$ and therefore $b = 1$. For a wave propagating in a bulk medium with the same refractive index ($n = n_2$) as the cladding (or for a wave propagating entirely in the cladding), $\beta = n_2 k_0$ and therefore $b = 0$.

The figure of b vs. V on the following page was scanned from Ref. [1]. The left-hand vertical axis of this figure shows b, which varies from 0 to 1. The right-hand vertical axis shows the **effective index**,

$$\overline{n} = \frac{\beta}{k_0}, \tag{130}$$

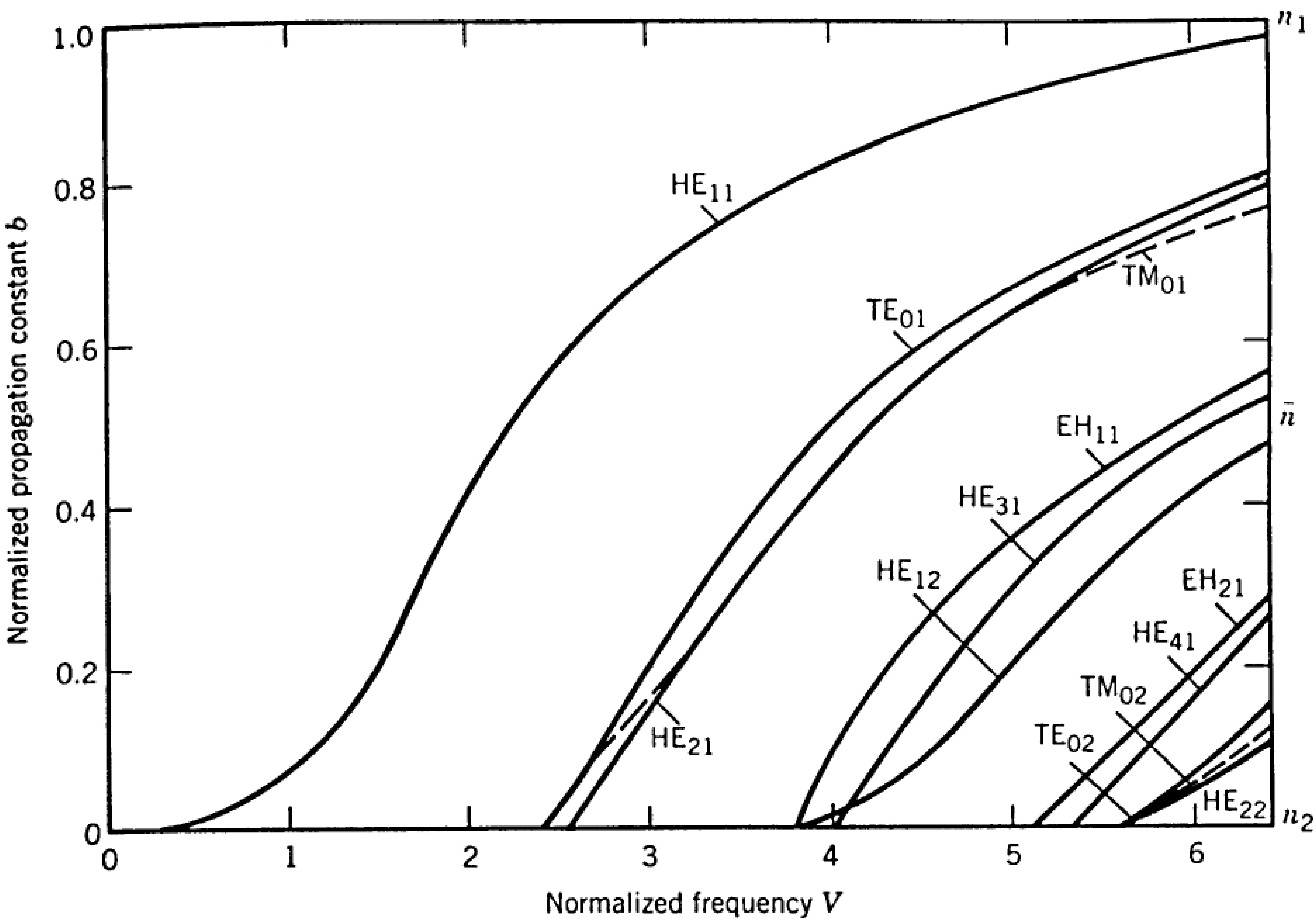
which varies from $\overline{n} = n_2$ when $b = 0$ to $\overline{n} = n_1$ when $b = 1$.

The figures of field lines on the pages following the figure of b vs. V were scanned from Ref. [2]. Solid lines represent lines of $\mathbf{E}$; dashed lines represent lines of $\mathbf{H}$.

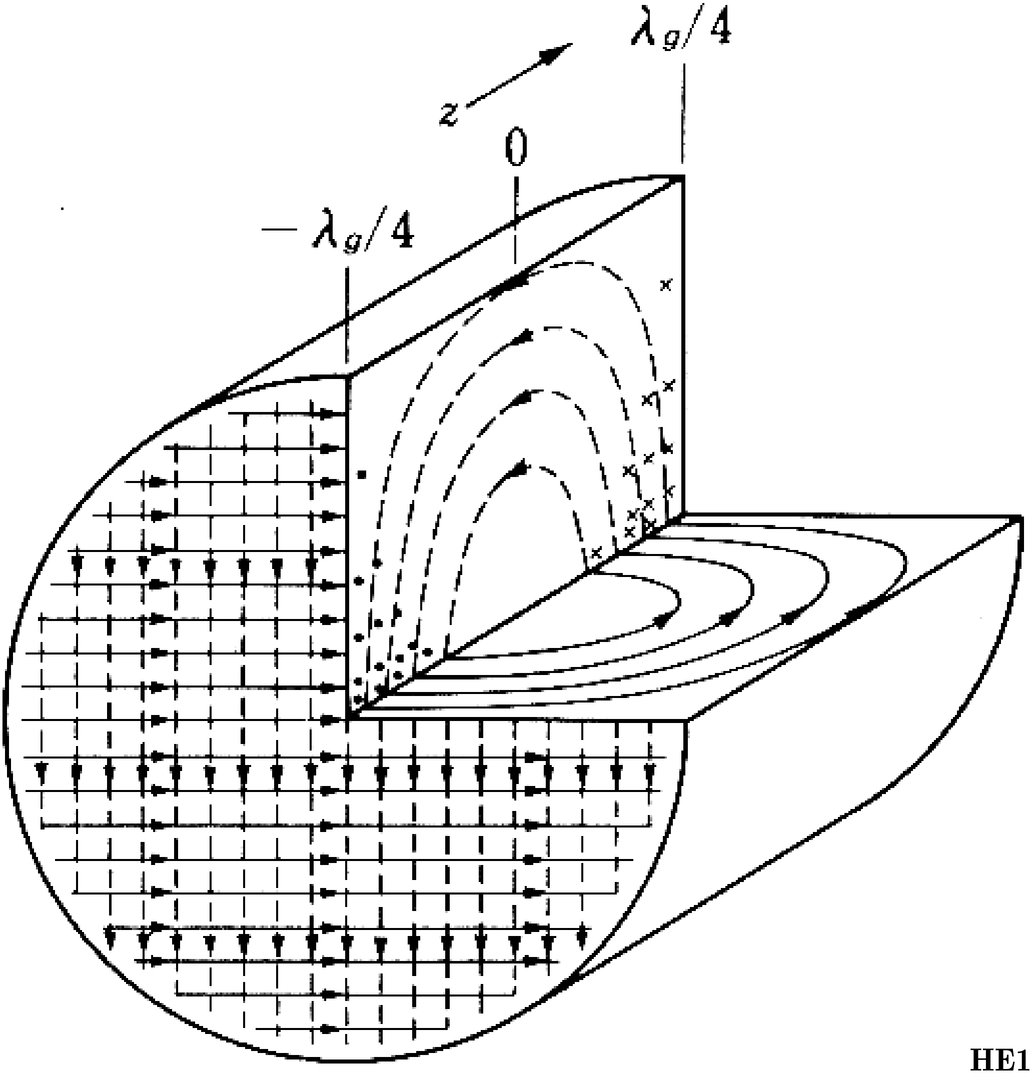
References

[1] Govind P. Agrawal, *Fiber-Optic Communication Systems*, Second Edition (New York, Wiley, 1997), p. 35.

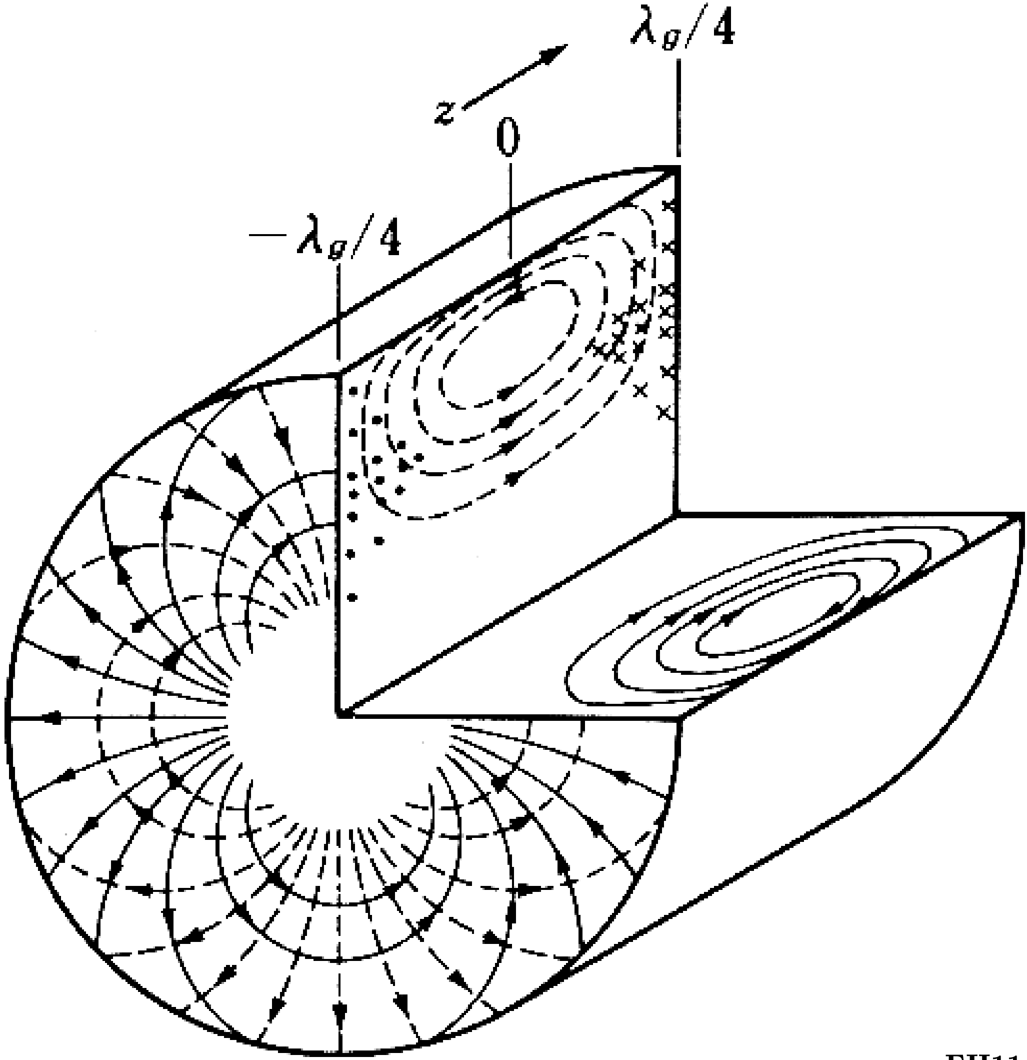
[2] Takanori Okoshi, *Optical Fibers* (New York, Academic Press, 1982), pp. 68–71.



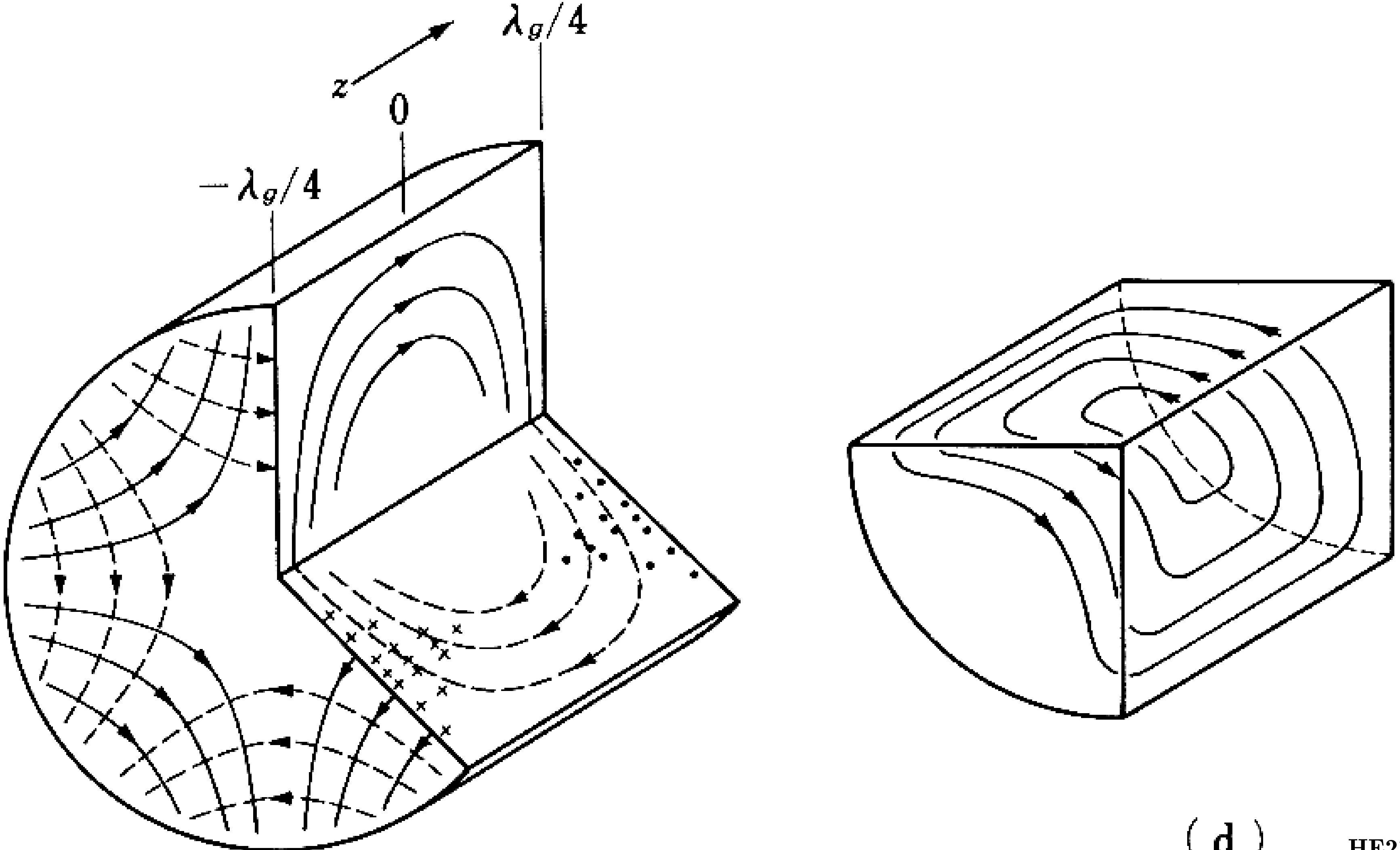
Normalized propagation constant b
Normalized frequency V
1.0
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
HE11
TE01
TM01
HE21
EH11
HE31
HE12
EH21
HE41
TM02
TE02
HE22
n1
n̄
n2



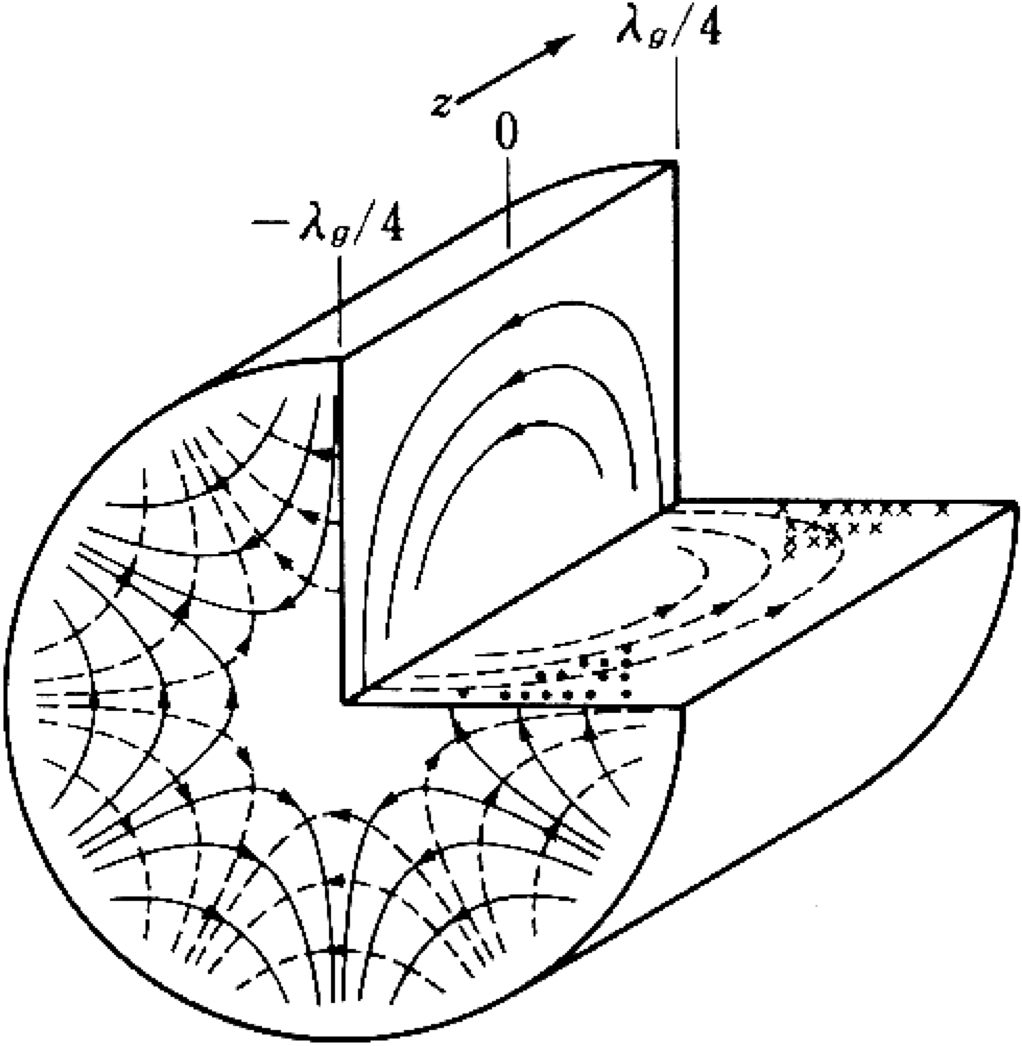
HE11



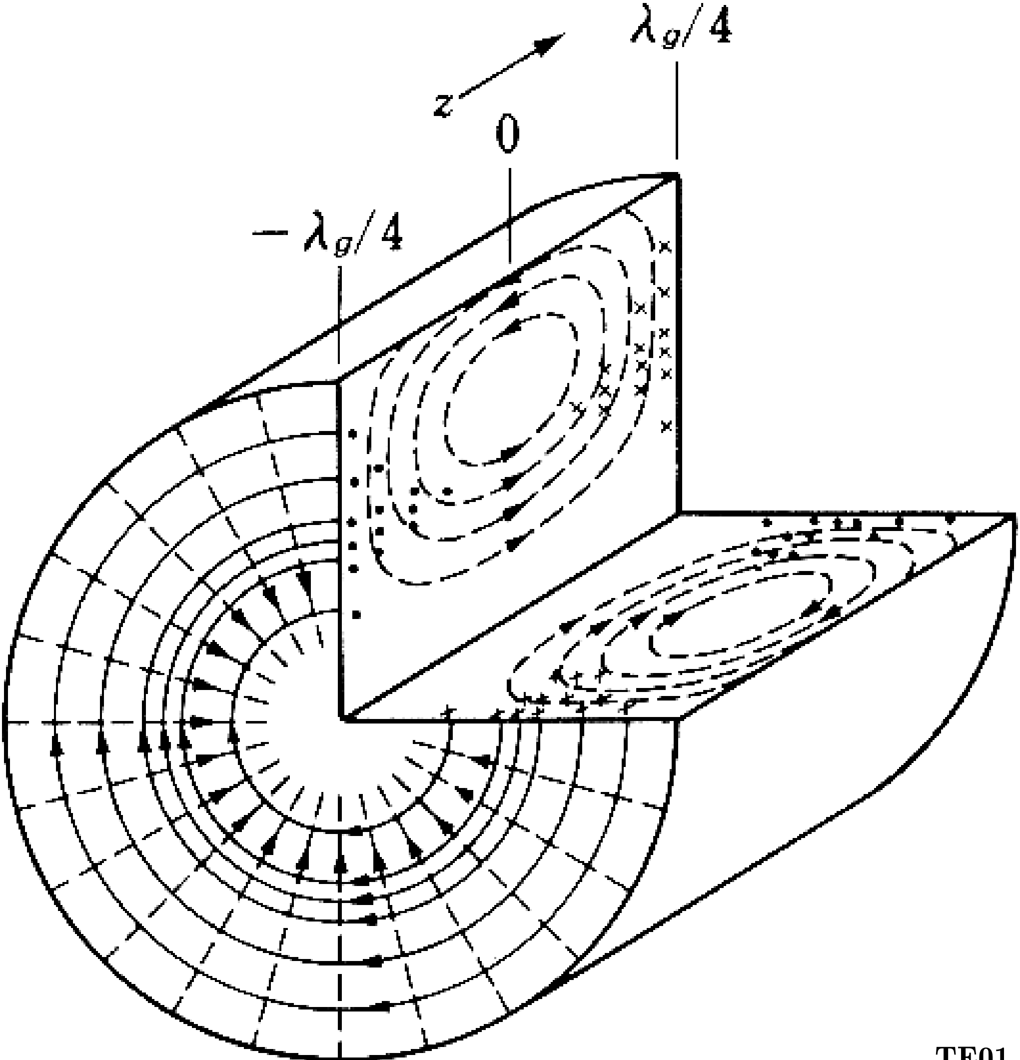
EH11



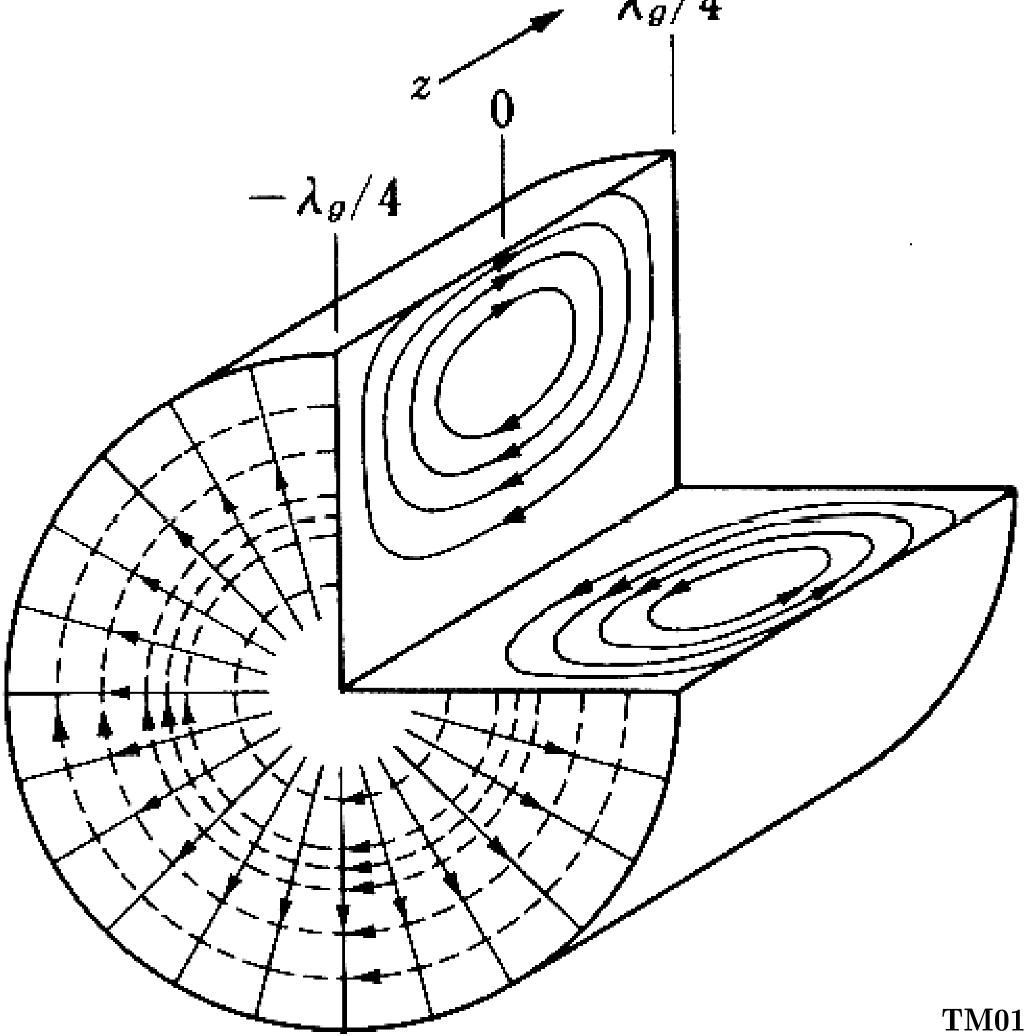
(d) HE21



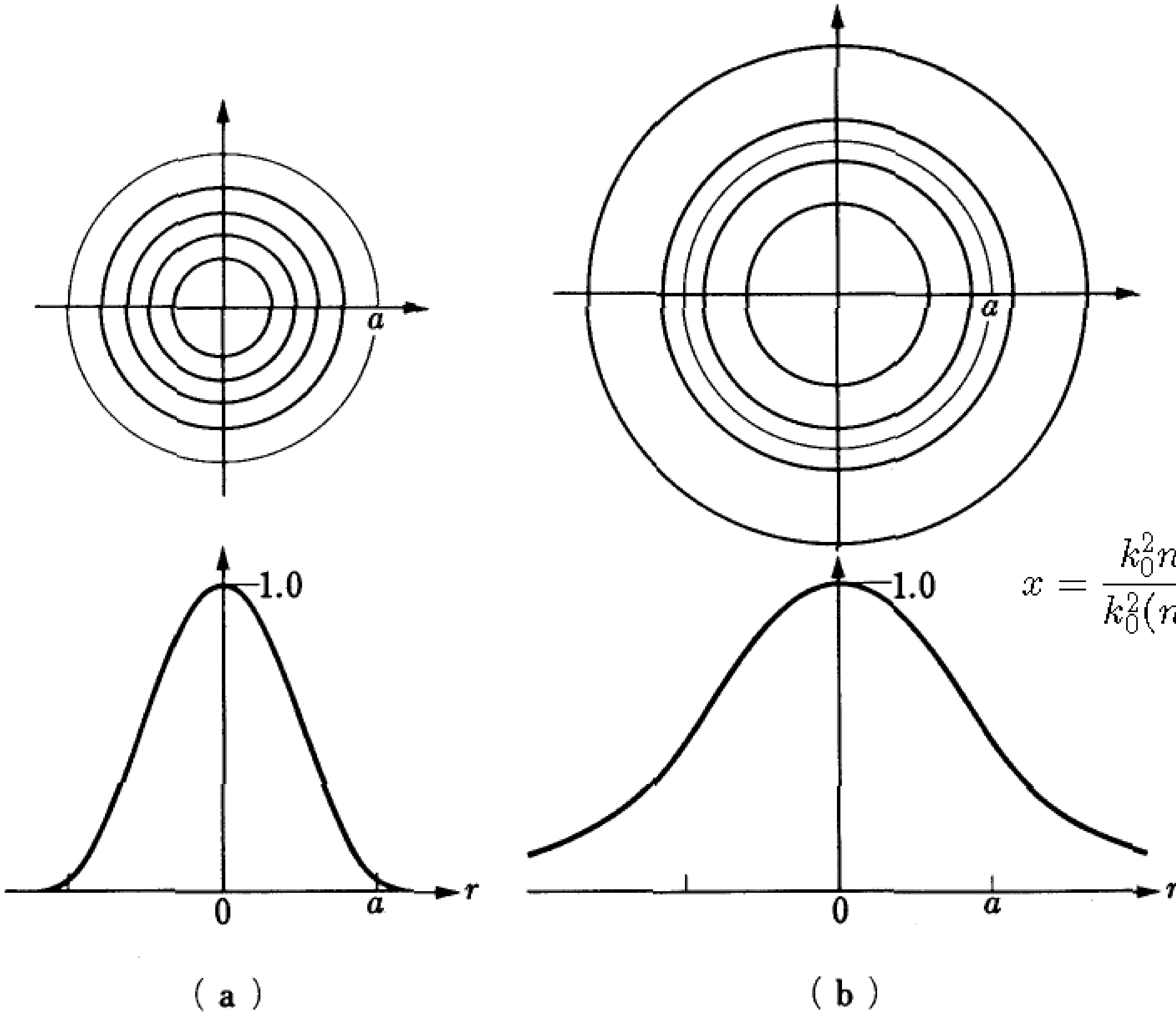
HE31



TE01

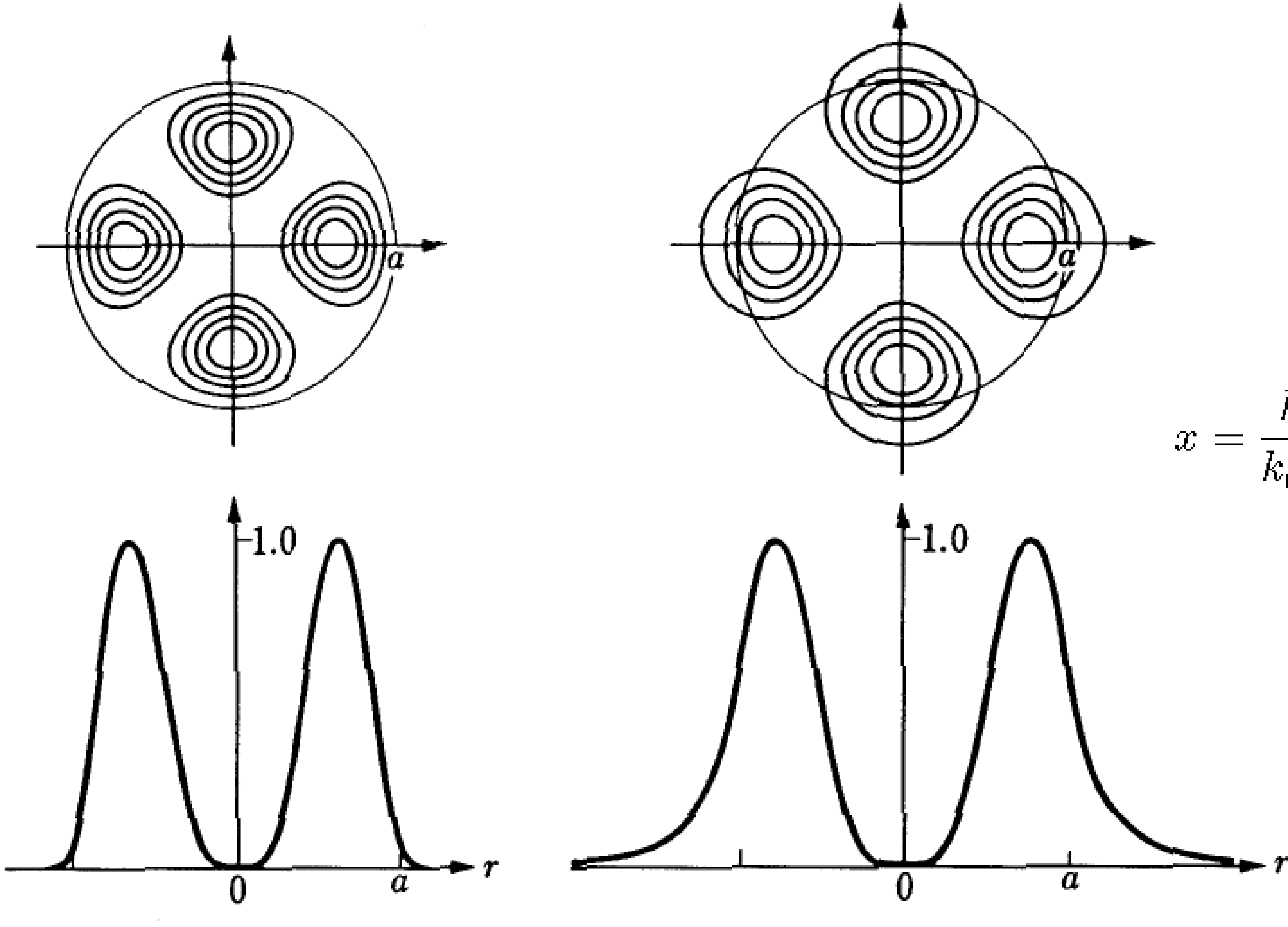


TM01



$$x = \frac{k_0^2 n_1^2 - \beta^2}{k_0^2 (n_1^2 - n_2^2)}$$

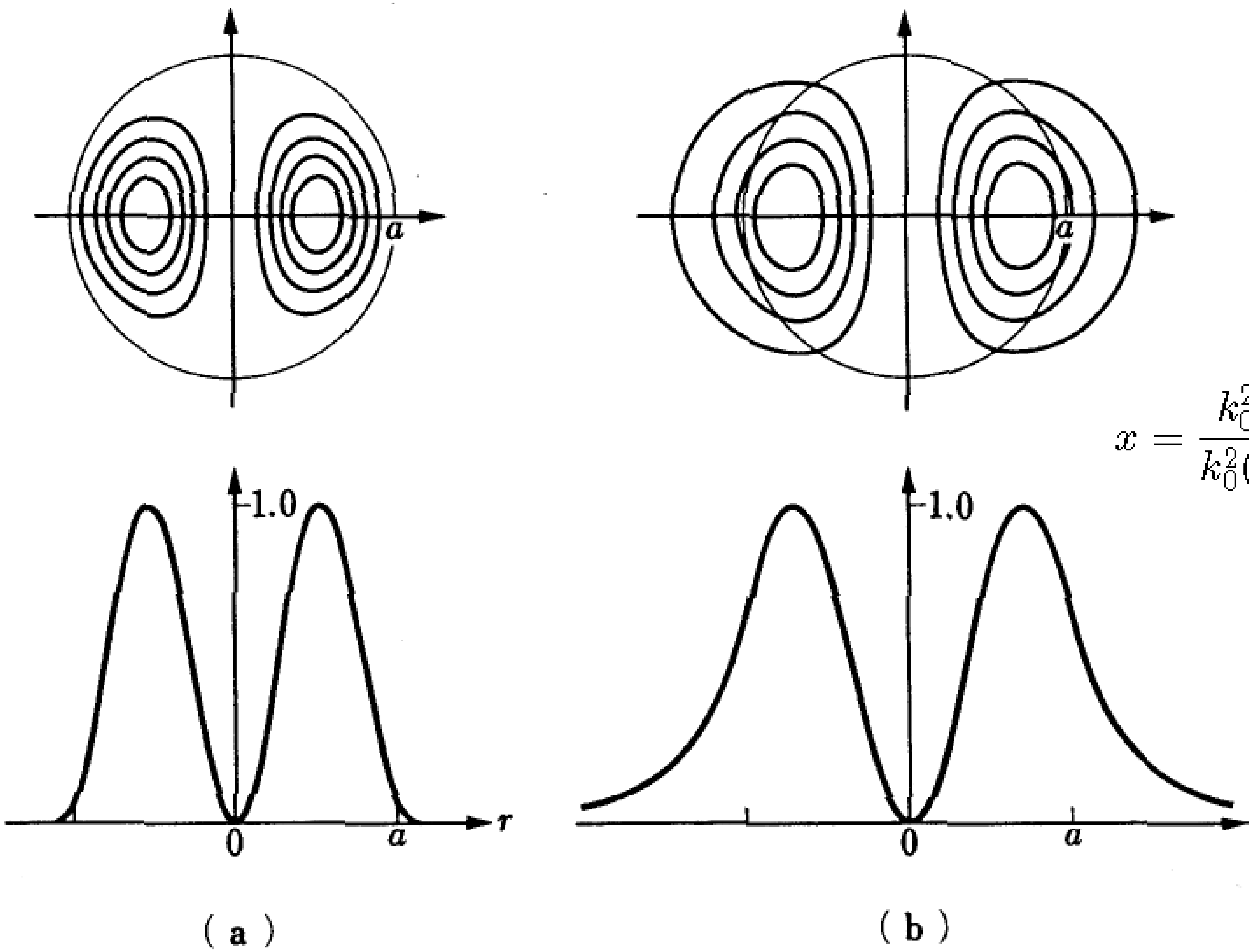
Fig. 4.6 Normalized power density distribution of the LP_{01} (HE_{11}) mode when (a) $x = 0.1$ and (b) $x = 0.9$.



$$x = \frac{k_0^2 n_1^2 - \beta^2}{k_0^2 (n_1^2 - n_2^2)}$$

(a) (b)

Fig. 4.8 Normalized power density distribution of the LP_{21} (EH_{11} and HE_{31}) mode when (a) $x = 0.1$ and (b) $x = 0.9$.



$$x = \frac{k_0^2 n_1^2 - \beta^2}{k_0^2 (n_1^2 - n_2^2)}$$

Fig. 4.7 Normalized power density distribution of the LP_{11} (TE_{01}, TM_{01}, and HE_{21}) mode when (a) $x = 0.1$ and (b) $x = 0.9$.