

## **FRESNEL-KIRCHHOFF DIFFRACTION THEORY**

- **Partial differential equation** approach:
  - ▷ Analytical solution
    - Solve PDE by separation of variables
    - Use boundary conditions to restrict set of possible solutions
    - Useful only for certain highly symmetrical geometries
  - ▷ Numerical solution:
    - Discretize using finite differences or finite elements
    - Obtain a system of linear equations that also incorporates the boundary conditions
- **Integral equation** approach:
  - ▷ Turn the PDE into an integral equation
  - ▷ Incorporate approximate boundary conditions
  - ▷ Get an approximate analytical solution
  - ▷ Numerical solution

## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (1)

- Maxwell's equations imply the vector wave equation

$$\nabla \times [\nabla \times \mathbf{E}(\mathbf{r}, t)] + \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) = 0$$

- ▷ Vector (polarization) effects are often negligible except for very small apertures or obstacles

- In a uniform dielectric and in Cartesian coordinates,

$$\left( \nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{E}(\mathbf{r}, t) = 0$$

- Scalar wave equation:

$$\left( \nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) U(\mathbf{r}, t) = 0$$

- ▷ Valid for (at best) one Cartesian component of  $\mathbf{E}$

## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (2)

- The scalar wave equation,

$$\left( \nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) U(\mathbf{r}, t) = 0,$$

and the assumption of a perfectly monochromatic wave,

$$U(\mathbf{r}, t) = u(\mathbf{r})e^{-i\omega t},$$

imply the scalar Helmholtz equation

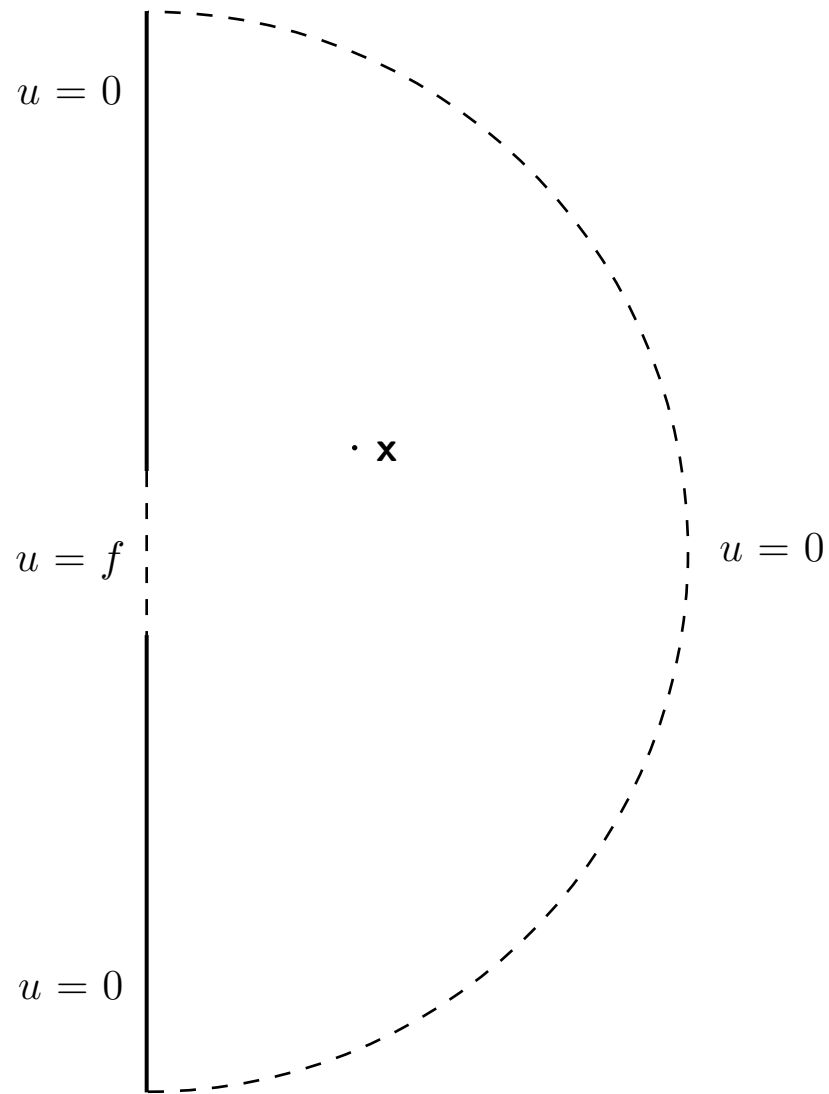
$$\boxed{(\nabla^2 + k^2) u(\mathbf{r}) = 0}$$

where

$$k^2 = \frac{\omega^2}{v^2}$$

- ▷ Boundary conditions for the Helmholtz equation
  - Dirichlet ( $u$  given on the boundary)
  - Neumann ( $du/dn$  given on the boundary)

# DIRICHLET BOUNDARY CONDITIONS



## BESSEL BEAMS

- Transverse radial profile is a Bessel function of the first kind
- **A Bessel beam propagates without spreading**
  - ▷ An exact solution of the Helmholtz equation  $(\nabla^2 + k^2)u = 0$  is

$$u(\rho, \phi, z) = J_m(\alpha\rho) e^{im\phi} e^{i\beta z}$$

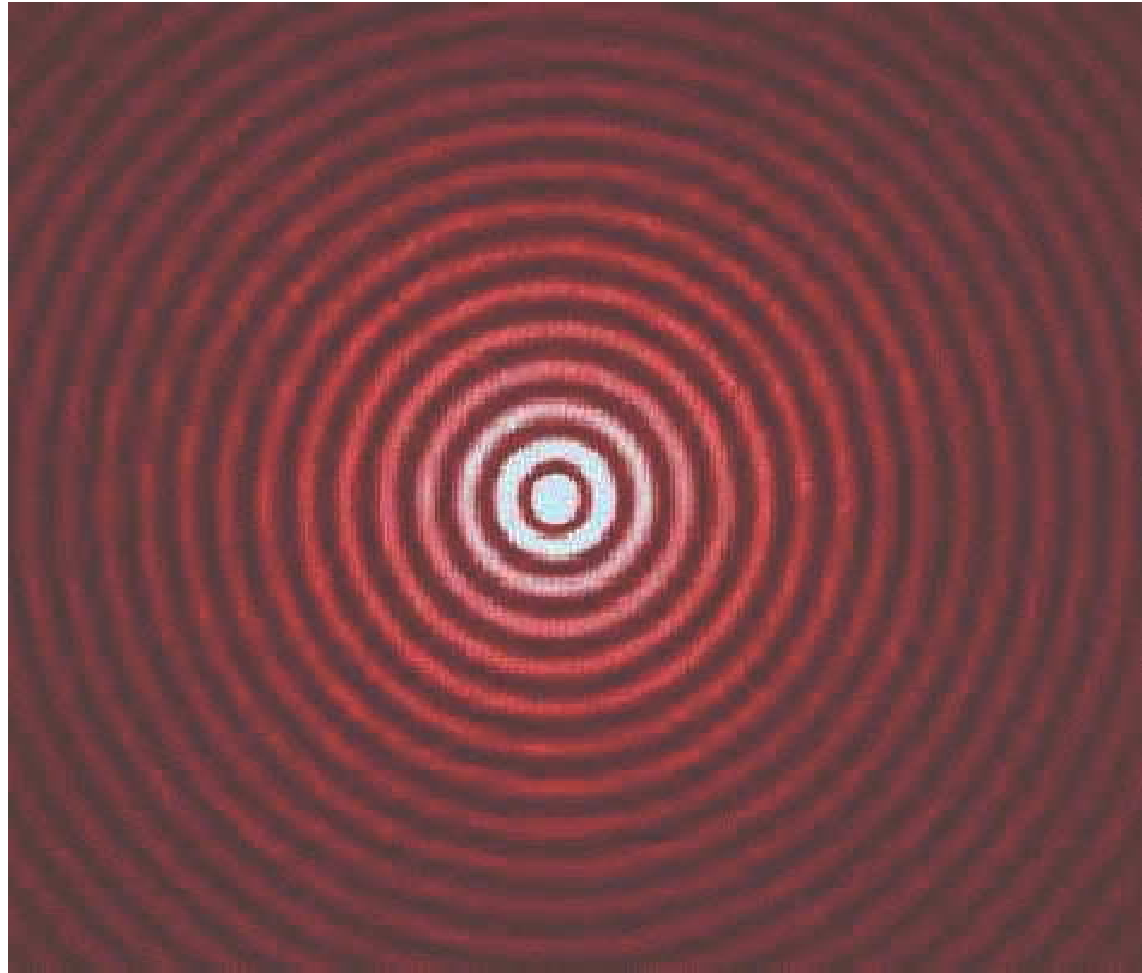
- The condition

$$k^2 = \alpha^2 + \beta^2$$

must be satisfied

- Most useful for  $m = 0$ , because  $J_0$  gives the beam a central bright spot
- Because the radial profile is independent of  $z$ , diffraction does not lead to spreading

# INTENSITY PATTERN OF A BESSEL BEAM



## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (3)

- Approximate scalar boundary conditions obtained from **St.-Venant's hypothesis**:
  - ▷ The optical field  $u$  in an aperture is the same as if the aperture were not present
  - ▷  $u = 0$ :
    - At all points on the screen
    - At large distances in image space
- Obviously not entirely correct, but useful
  - ▷ Ignores currents in the edges of the aperture

## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (4)

- Green function for the scalar Helmholtz equation:

▷ Scalar Helmholtz equation:

$$(\nabla^2 + k^2)u = 0$$

▷ Green function in free space:

$$G(\mathbf{r}, \mathbf{x}) = \frac{e^{ik|\mathbf{r}-\mathbf{x}|}}{|\mathbf{r}-\mathbf{x}|}$$

- $G$  is a spherical wave expanding out from a point source at  $\mathbf{r}$
- $G$  satisfies the equation

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{x}) = -4\pi\delta(\mathbf{r} - \mathbf{x})$$

and outgoing boundary conditions at  $\infty$

▷ A solution of the equation  $(\nabla^2 + k^2)u(\mathbf{r}) = -s(\mathbf{r})$  in a region  $V$  is

$$u(\mathbf{x}) = \begin{cases} (4\pi)^{-1} \int_V G(\mathbf{x} - \mathbf{r}) s(\mathbf{r}) d^3r, & \text{if } \mathbf{x} \in V; \\ 0, & \text{if } \mathbf{x} \notin V \end{cases}$$

**FRESNEL-KIRCHHOFF DIFFRACTION THEORY (5)**

- Green function for the scalar Helmholtz equation:

▷ Scalar Helmholtz equation:

$$(\nabla^2 + k^2)u = 0$$

▷ Green function in free space (outgoing boundary conditions):

$$G(\mathbf{r}, \mathbf{x}) = \frac{e^{ik|\mathbf{r}-\mathbf{x}|}}{|\mathbf{r}-\mathbf{x}|}$$

- $G$  is a spherical wave expanding out from a point source at  $Q$
- $G$  satisfies the equation

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{x}) = -4\pi\delta(\mathbf{r} - \mathbf{x})$$

and outgoing boundary conditions at  $\infty$

▷ A solution of the equation  $(\nabla^2 + k^2)u(\mathbf{r}) = -s(\mathbf{r})$  is

$$u(\mathbf{x}) = \frac{1}{4\pi} \int G(\mathbf{x} - \mathbf{r}) s(\mathbf{r}) d^3r$$

## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (6)

- Derivation of Green's theorem:

▷ Recall Gauss's theorem:

$$\int_V \nabla \cdot \mathbf{F} d^3r = \int_{\partial V} (-\hat{\mathbf{n}}) \cdot \mathbf{F} dS$$

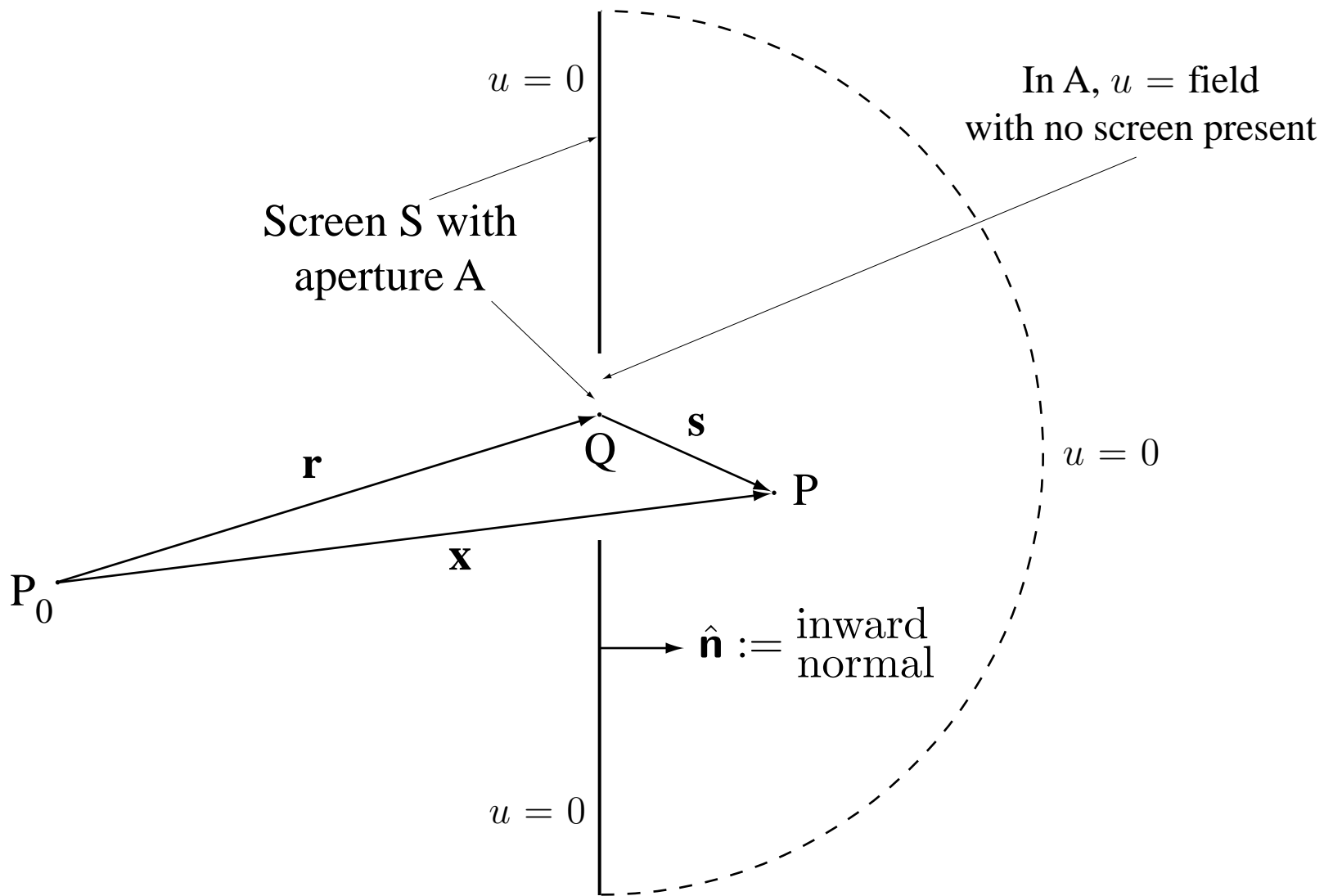
$\partial V$  := surface that bounds the volume  $V$ ,  $\hat{\mathbf{n}}$  := INWARD normal to  $\partial V$

▷ Apply Gauss's theorem to the vector field  $\mathbf{F} = u\nabla G - G\nabla u$ :

$$\begin{aligned} \int_{\partial V} (-\hat{\mathbf{n}}) \cdot (u\nabla G - G\nabla u) dS &= \int_V \nabla \cdot (u\nabla G - G\nabla u) d^3r \\ &= \int_V [(\nabla u) \cdot (\nabla G) - (\nabla G) \cdot (\nabla u)] d^3r \\ &\quad + \int_V (u\nabla^2 G - G\nabla^2 u) d^3r \\ &= \int_V [u(\nabla^2 + k^2)G - G(\nabla^2 + k^2)u] d^3r = -4\pi u(\mathbf{x}) \end{aligned}$$

- We used  $(\nabla^2 + k^2)u = 0$  and  $(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{x}) = -4\pi\delta(\mathbf{r} - \mathbf{x})$

# KIRCHHOFF'S BOUNDARY CONDITIONS



## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (7)

- The **Helmholtz-Kirchhoff integral theorem**:

▷ From Gauss's and Green's theorems,

$$u(\mathbf{x}) = \frac{1}{4\pi} \int_{\partial V} [u(\hat{\mathbf{n}} \cdot \nabla G) - G(\hat{\mathbf{n}} \cdot \nabla u)] dS$$

$\hat{\mathbf{n}} :=$  INWARD normal to the surface that bounds  $V$

$$G(\mathbf{r}, \mathbf{x}) = \frac{e^{ik|\mathbf{r}-\mathbf{x}|}}{|\mathbf{r}-\mathbf{x}|}$$

▷ Substitute for  $G$ :

$$u(\mathbf{x}) = \frac{1}{4\pi} \int_{\partial V} \left[ u(\mathbf{r}) \left( \hat{\mathbf{n}} \cdot \nabla \frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} (\hat{\mathbf{n}} \cdot \nabla u(\mathbf{r})) \right] dS$$

- Expresses  $u(\mathbf{x})$  in terms of fields at points  $\mathbf{r}$  on the surface that bounds  $V$ , but not in the same way as Huyghens' Principle
- Kirchhoff's boundary conditions apply

## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (8)

- Kirchhoff's boundary conditions:

▷  $u = 0$  everywhere but in the aperture

▷ In the aperture,  $u$  is the field due to a point source at  $P_0$ ,

$$u(\mathbf{r}) = A \frac{e^{ikr}}{r}$$

- Apply the Helmholtz-Kirchhoff integral theorem:

▷ At any point  $\mathbf{r}$  in the aperture,

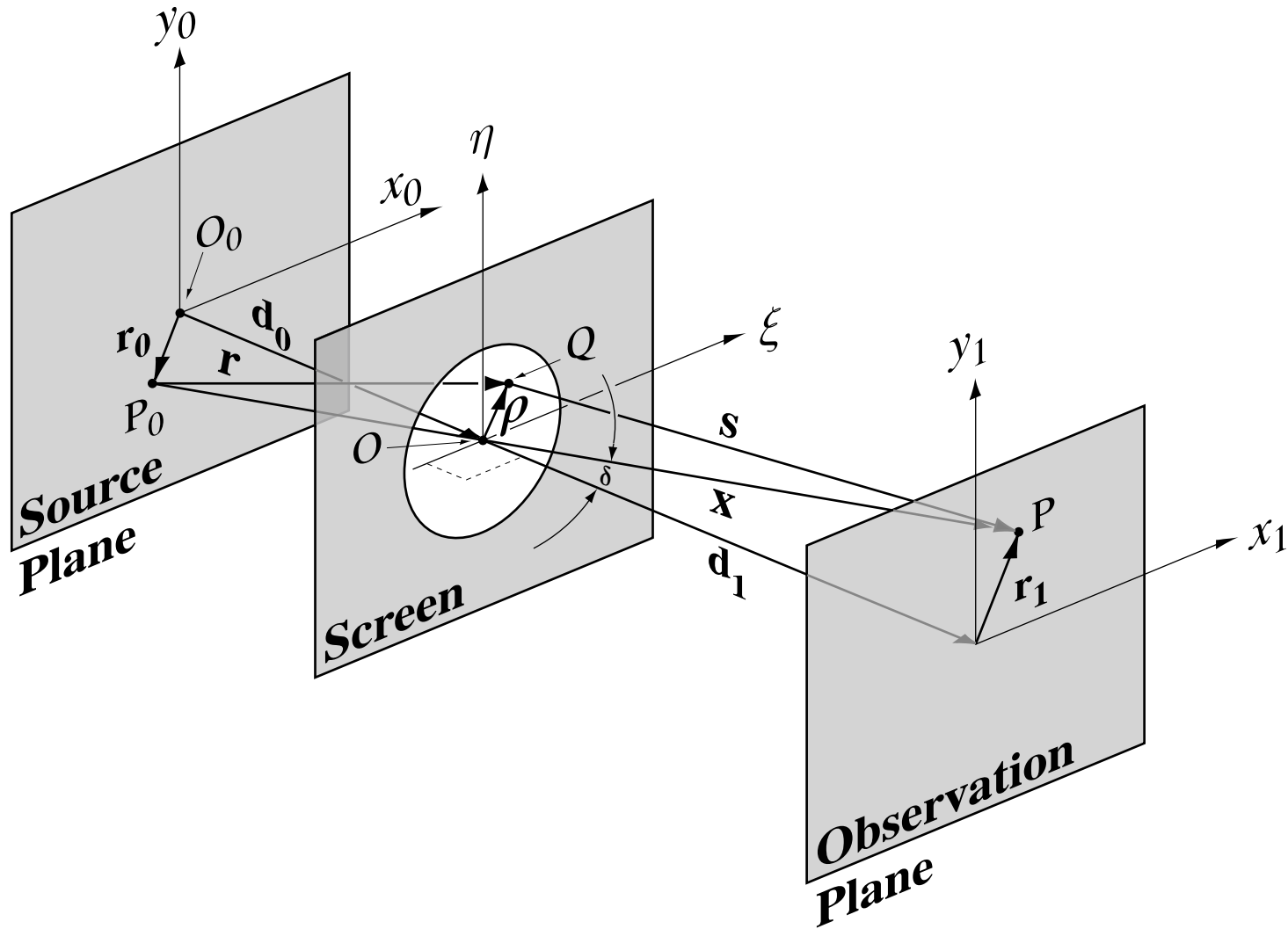
$$\nabla u(\mathbf{r}) = \hat{\mathbf{r}} A \frac{e^{ikr}}{r} \left( ik - \frac{1}{r} \right), \quad \nabla G(\mathbf{r}, \mathbf{x}) = -\hat{\mathbf{s}} A \frac{e^{iks}}{s} \left( ik - \frac{1}{s} \right)$$

where  $\mathbf{s} := \mathbf{x} - \mathbf{r}$

▷ Result: The **Fresnel-Kirchhoff diffraction formula**,

$$u(\mathbf{x}) = -\frac{ikA}{4\pi} \int_A \frac{e^{ik(r+s)}}{rs} (\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}}) dS$$

# COORDINATES FOR THE FRESNEL-KIRCHHOFF DIFFRACTION INTEGRAL



## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (9)

- The **Fresnel-Kirchhoff diffraction formula**

$$u(\mathbf{x}) = -\frac{ikA}{4\pi} \int_A \frac{e^{ik(r+s)}}{rs} (\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}}) dS$$

- ▷ is an approximate scalar solution of the Helmholtz equation, valid for wavelengths small compared to the aperture size
- ▷ is based on deep physical insight
- ▷ is accurate enough to give useful, quantitative results in optics
- BUT
- ▷ does not make  $u \rightarrow 0$  at large distances (as assumed in Kirchhoff's boundary conditions)
- ▷ is not the first term in any iterative solution of the boundary-value problem

## FRESNEL-KIRCHHOFF DIFFRACTION THEORY (10)

- Consistency of the **Fresnel-Kirchhoff diffraction formula**

$$u(\mathbf{x}) = -\frac{ikA}{4\pi} \int_A \frac{e^{ik(r+s)}}{rs} (\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}}) dS$$

with Huyghens' Principle

- ▷ Expresses  $u(\mathbf{x})$  as a superposition of spherical waves  $e^{iks}/s$  emanating from the wavefront  $e^{ikr}/r$  produced by a point source
- ▷ The **inclination factor** is

$$I(\hat{\mathbf{r}}, \hat{\mathbf{s}}) = -(\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}})$$

- In the backward direction ( $\hat{\mathbf{s}} = -\hat{\mathbf{r}}$ ),

$$I(\hat{\mathbf{r}}, -\hat{\mathbf{r}}) = 0$$

- In the forward direction ( $\hat{\mathbf{s}} = \hat{\mathbf{r}}$ ),

$$I(\hat{\mathbf{r}}, \hat{\mathbf{r}}) = -2 \cos \delta$$

**FRESNEL-KIRCHHOFF DIFFRACTION THEORY (11)**

- Extensions of the Fresnel-Kirchhoff diffraction formula:

▷ If the inclination factor is nearly constant over the aperture,

$$u(\mathbf{x}) \approx \frac{ikAI}{4\pi} \int_A \frac{e^{ik(r+s)}}{rs} dS$$

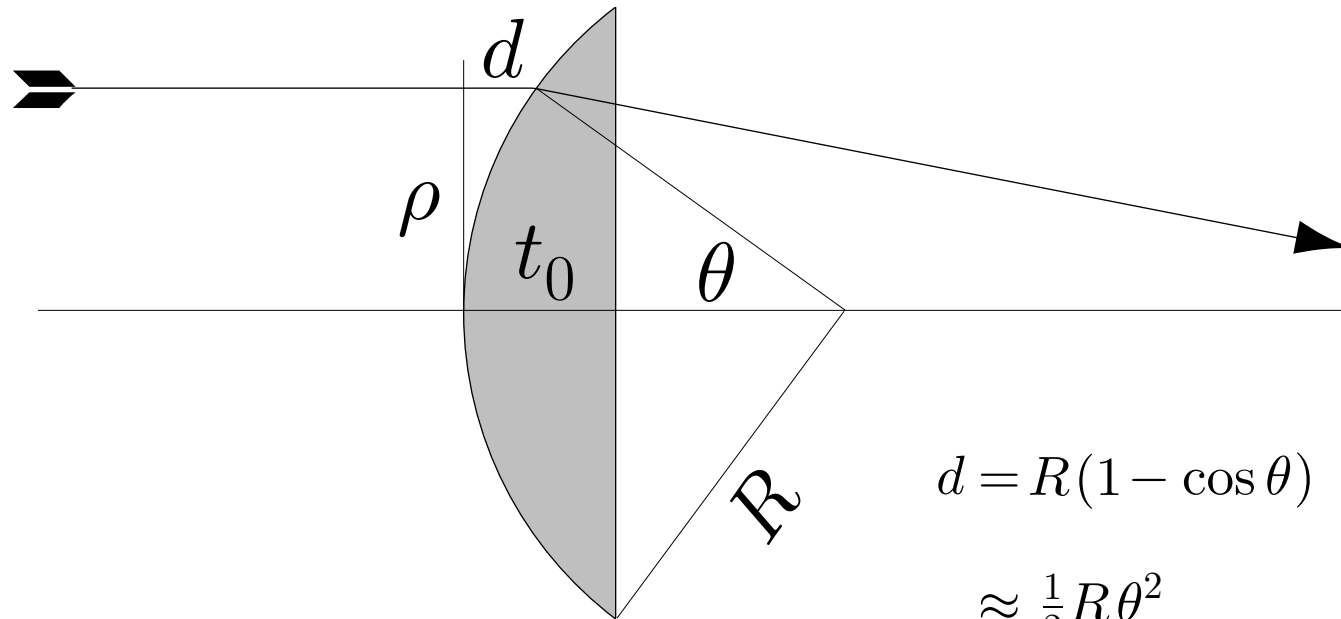
▷ Replace the point-source incident wavefront  $e^{ikr}/r$  produced by a point source with a general wavefront  $u_{\text{inc}}(\mathbf{r})$  produced by an extended source (modeled as a collection of point sources):

$$u(\mathbf{x}) = \frac{ikI}{4\pi} \int_A u_{\text{inc}}(\mathbf{r}) \frac{e^{iks}}{s} dS$$

▷ Characterize the aperture by a **transmission function**  $\tau$  to model amplitude and/or phase changes due to lenses & diffraction gratings

$$u(\mathbf{x}) = \frac{ikI}{4\pi} \int_A \tau(\mathbf{r}) u_{\text{inc}}(\mathbf{r}) \frac{e^{iks}}{s} dS$$

# THIN LENS



$$d = R(1 - \cos \theta)$$

$$\approx \frac{1}{2}R\theta^2$$

$$\approx \frac{\rho^2}{2R}$$

$$\text{Optical path in lens} \approx kn \left[ t_0 - \frac{\rho^2}{2R} \right]$$

## TRANSMISSION FUNCTION OF A THIN LENS

- Transmission function of a thin lens (thickness =  $t$ ):

$$\tau(\boldsymbol{\rho}) \approx |\tau(\boldsymbol{\rho})| e^{iknt} e^{-ik\rho^2/(2f)}$$

- ▷ If the incident field is  $u_{\text{inc}}(\mathbf{r}) = e^{ikz}$ , then the field transmitted by a plano-convex thin lens (neglecting reflections) is

$$u_{\text{tr}}(\mathbf{r}) \approx \exp ik \left\{ z + \frac{\rho^2}{2R} + n \left[ t_0 - \frac{\rho^2}{2R} \right] \right\}$$

- ▷ Therefore the plano-convex transmission function is

$$\tau_{\text{plano}}(\boldsymbol{\rho}) \approx e^{iknt_0} e^{-ik(n-1)\rho^2/(2R)}$$

- ▷ For a biconvex lens, with curvatures  $R_1$  and  $-R_2$ ,

$$\tau_{\text{biconvex}}(\boldsymbol{\rho}) \approx e^{ikn(t_1+t_2)} e^{-ik(n-1)\rho^2/[(2R_1)^{-1} - (2R_2)^{-1}]}$$

- ▷ Focal length  $f =$  net curvature: 
$$\frac{1}{f} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

## DIFFRACTION THEORY (1)

- Applications of the Fresnel-Kirchhoff diffraction formula

$$u(\mathbf{x}) = \frac{ikI}{4\pi} \int_A \tau(\mathbf{r}) u_{\text{inc}}(\mathbf{r}) \frac{e^{iks}}{s} dS$$

- ▷ Diffraction of light from a point source:

$$u_{\text{inc}}(\mathbf{r}) = A \frac{e^{ikr}}{r}$$

- Fraunhofer limit (plane waves)
- Fresnel regime
- ▷ Diffraction theory of image formation
  - Aberrations
  - Focal-plane filtering
  - Transfer functions
- ▷ Propagation of the mutual coherence function

## DIFFRACTION THEORY (2)

- Fresnel's approximation to a spherical wave:

$$\frac{e^{ikr}}{r} \approx \frac{e^{ikr'}}{r'} e^{i\mathbf{k}_1 \cdot \boldsymbol{\rho}} e^{ik[\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2]/(2r')}$$

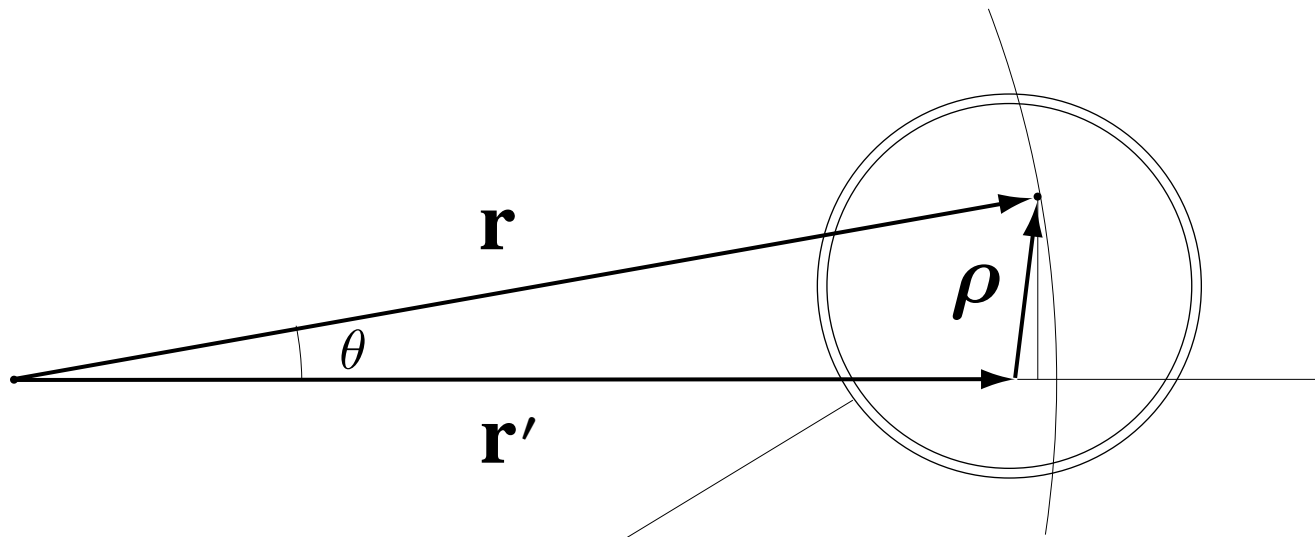
- ▷  $\frac{e^{ikr'}}{r'}$  = spherical wave centered at  $P_0$
- ▷  $e^{i\mathbf{k}_1 \cdot \boldsymbol{\rho}}$  = plane wave with propagation constant  $\mathbf{k}_1 := k\hat{\mathbf{r}}'$
- ▷  $e^{ik[\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2]/(2r')}$  = lowest correction for wavefront curvature
  - This factor is approximately 1 when the Fresnel number is small:

$$\frac{b^2}{r'\lambda} \ll 1$$

where  $b^2 := \rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2$

- Bottom line: You get a nearly plane wave if you move far away from a point source (but it's easier to use a lens!)

# THE FRESNEL APPROXIMATION



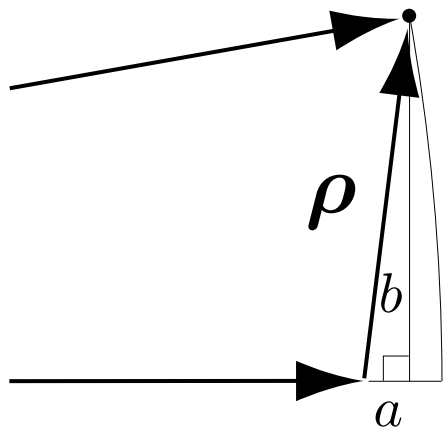
$$a = \hat{\mathbf{r}}' \cdot \boldsymbol{\rho}$$

$$b^2 = \rho^2 - a^2 = \rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2$$

$$r = \frac{r' + a}{\cos \theta}$$

$$\approx \frac{r'}{\cos \theta} + a \approx r'(1 + \frac{1}{2}\theta^2) + a$$

$$\approx r' + a + \frac{b^2}{2r'}$$



## FRESNEL APPROXIMATION DETAILS (1)

- Approximate  $r$  in terms of  $r'$  (fixed) and  $\rho$  (varies over aperture):

▷ Vector addition:

$$\begin{aligned}\mathbf{r} &= \mathbf{r}' + \boldsymbol{\rho} \\ r^2 &= r'^2 + 2\mathbf{r}' \cdot \boldsymbol{\rho} + \rho^2 \\ r &= r' \left[ 1 + 2\frac{\mathbf{r}' \cdot \boldsymbol{\rho}}{r'^2} + \left(\frac{\rho}{r'}\right)^2 \right]^{1/2}\end{aligned}$$

▷ Binomial expansion correct to order  $(\rho/r')^2$ :

$$\begin{aligned}r &\approx r' + \frac{\mathbf{r}' \cdot \boldsymbol{\rho}}{r'} + \frac{\rho^2}{2r'} - \frac{(\mathbf{r}' \cdot \boldsymbol{\rho})^2}{2r'^3} \\ r &\approx r' + \hat{\mathbf{r}}' \cdot \boldsymbol{\rho} + \frac{\rho^2}{2r'} - \frac{(\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2}{2r'}\end{aligned}$$

## FRESNEL APPROXIMATION DETAILS (2)

- The **Fresnel approximation**, valid for small angles between  $\mathbf{r}$  and  $r'$ :

$$\text{denominator} = r \approx r'$$

$$\text{phase} = kr \approx kr' + \mathbf{k}_1 \cdot \boldsymbol{\rho} + k \left[ \frac{\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2}{2r'} \right]$$

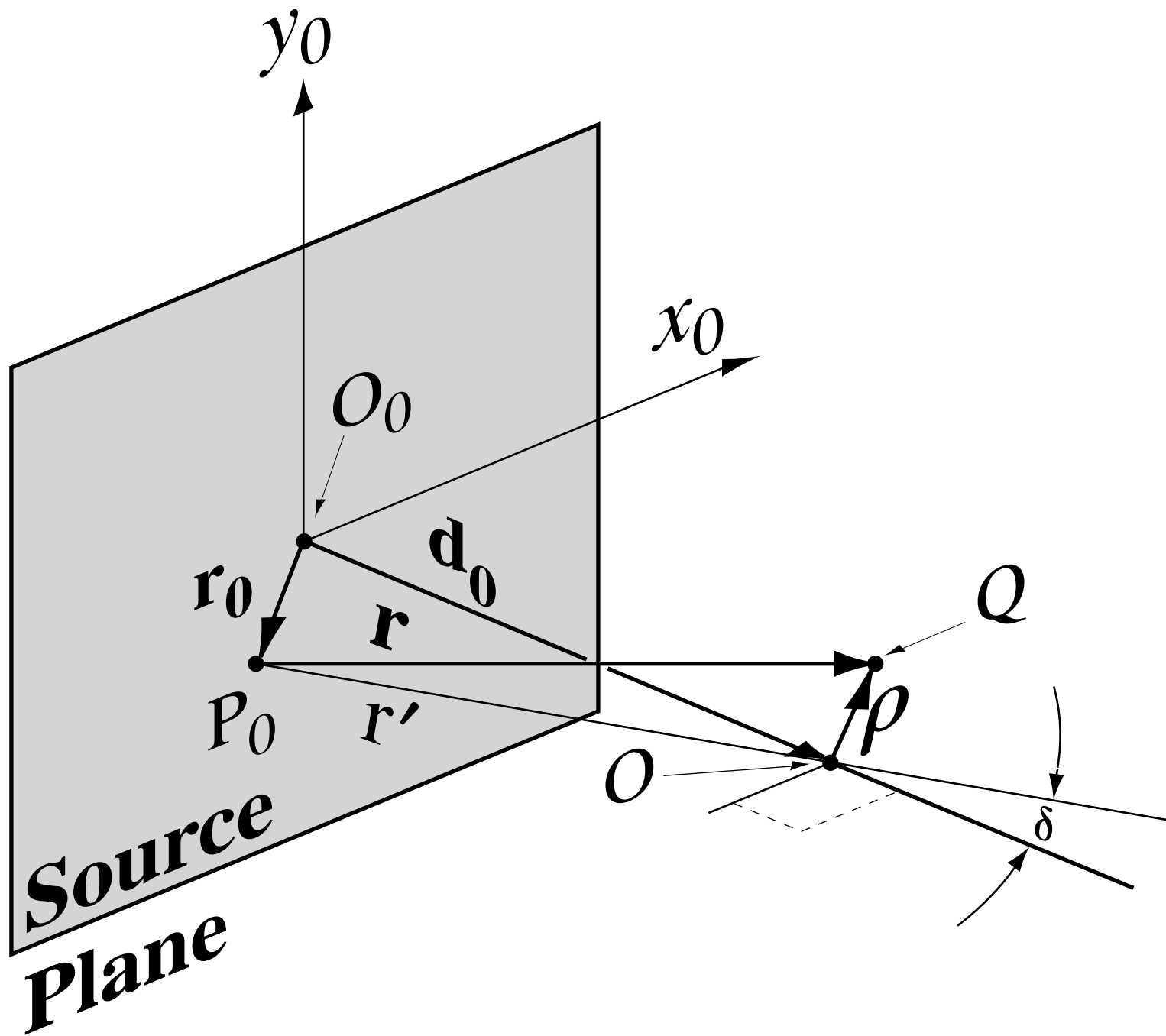
▷  $\frac{e^{ikr'}}{r'}$  = spherical wave centered at  $P_0$

▷  $e^{i\mathbf{k}_1 \cdot \boldsymbol{\rho}}$  = plane wave with direction  $\hat{\mathbf{r}}'$

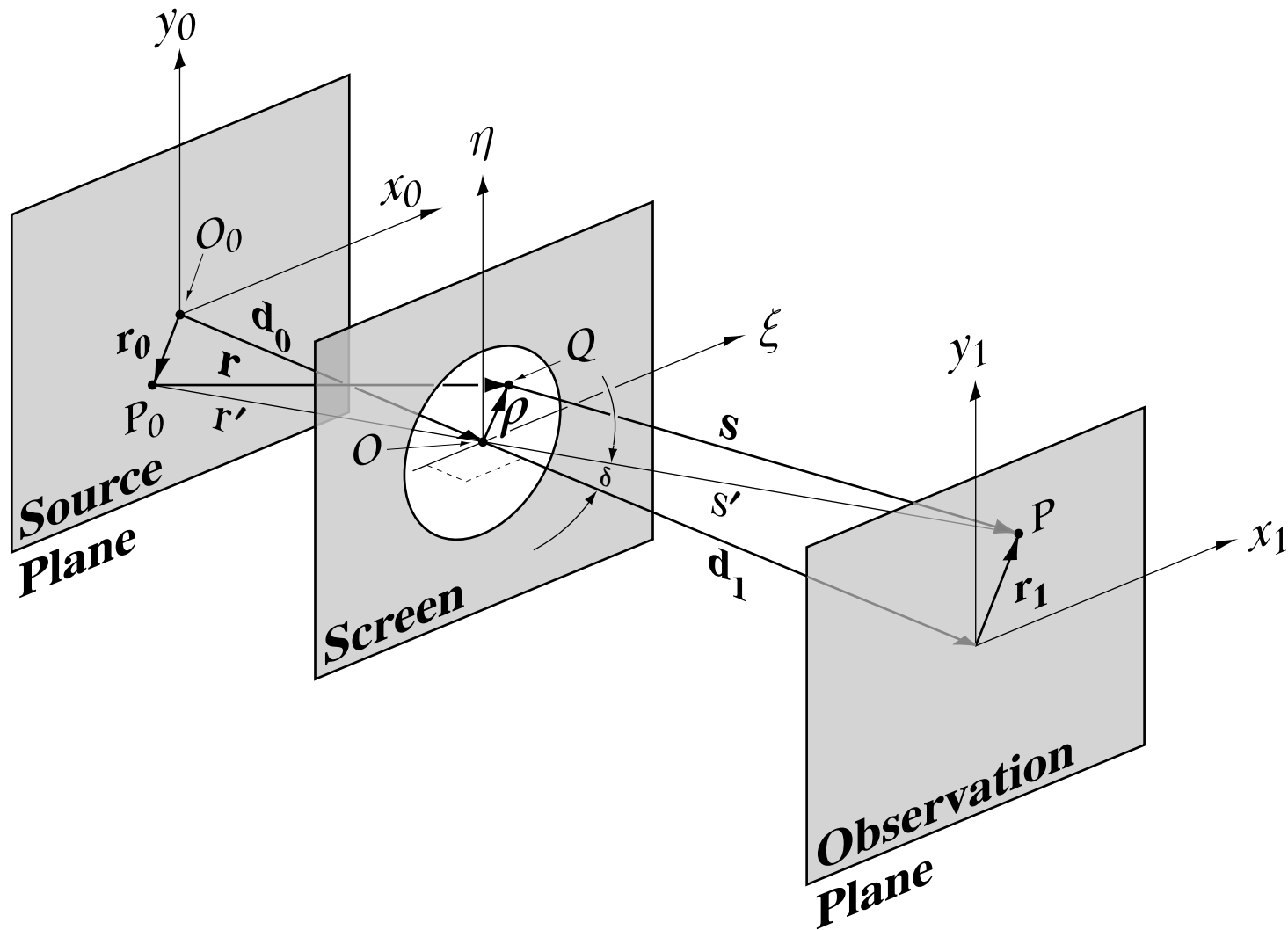
◦ Propagation constant:  $\mathbf{k}_1 := k\hat{\mathbf{r}}'$

▷  $e^{ik[\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2]/(2r')}$  = lowest-order approximation to the phase of a spherical wavefront

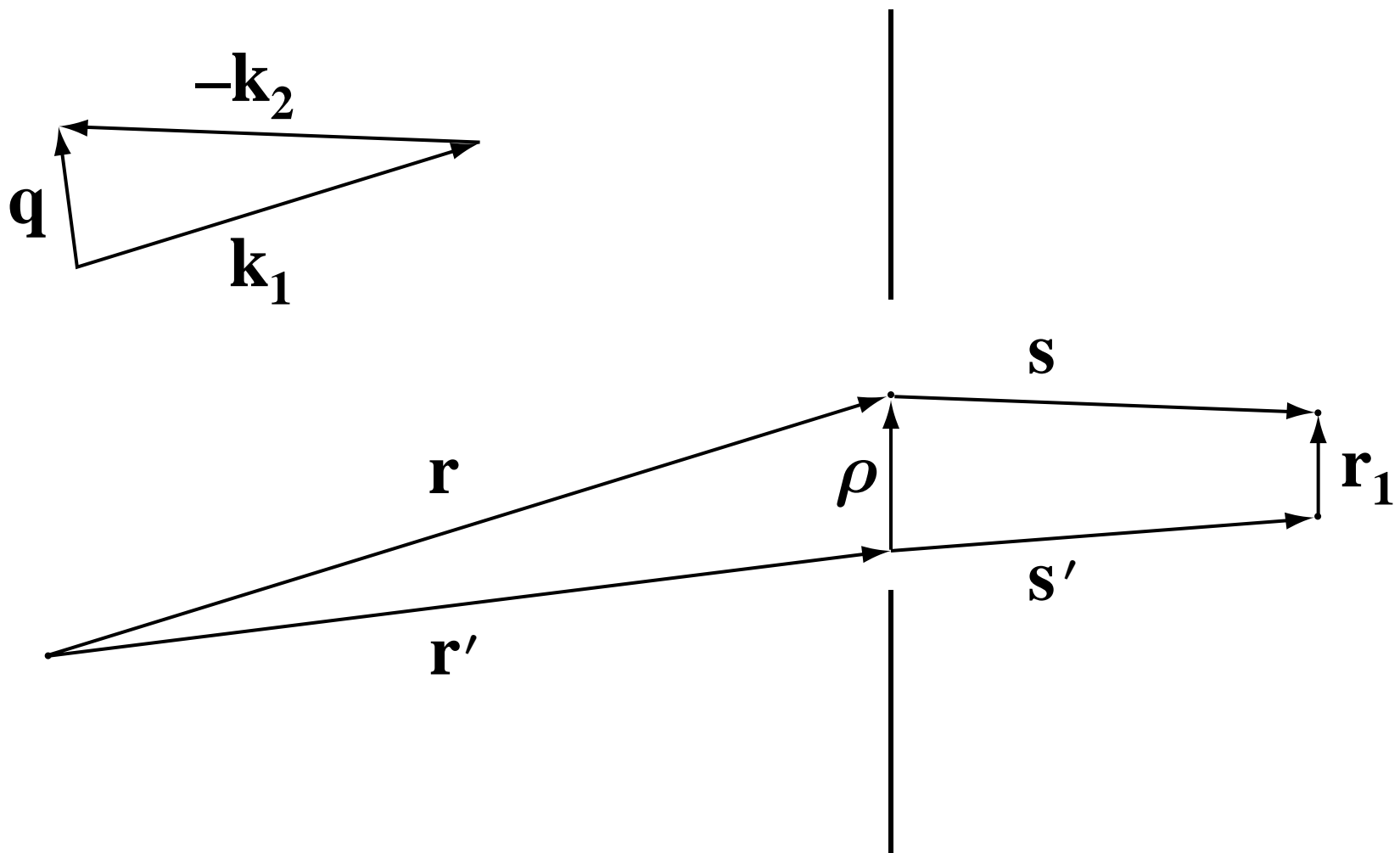
$$\frac{e^{ikr}}{r} \approx \frac{e^{ikr'}}{r'} e^{i\mathbf{k}_1 \cdot \boldsymbol{\rho}} e^{ik[\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2]/(2r')}$$



# COORDINATES FOR THE FRESNEL-KIRCHHOFF DIFFRACTION INTEGRAL



# COORDINATES FOR THE DIFFRACTION INTEGRAL



## DIFFRACTION THEORY (3)

- Diffraction pattern of a point source, part 1:

▷ Fresnel's approximation applied to the spherical waves  $e^{ikr}/r$  and  $e^{iks}/s$ :

$$\frac{e^{ikr}}{r} \approx \frac{e^{ikr'}}{r'} e^{i\mathbf{k}_1 \cdot \boldsymbol{\rho}} \exp \left\{ ik \left[ \frac{\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2}{2r'} \right] \right\}$$

$$\frac{e^{iks}}{s} \approx \frac{e^{iks'}}{s'} e^{i\mathbf{k}_2 \cdot (\mathbf{r}_1 - \boldsymbol{\rho})} \exp \left\{ ik \left[ \frac{(\mathbf{r}_1 - \boldsymbol{\rho})^2 - [\hat{\mathbf{s}}' \cdot (\mathbf{r}_1 - \boldsymbol{\rho})]^2}{2s'} \right] \right\}$$

where the propagation vectors are

$$\mathbf{k}_1 := k\hat{\mathbf{r}}', \quad \mathbf{k}_2 := k\hat{\mathbf{s}}', \quad \mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2$$

▷ Diffraction integral for a point source in Fresnel's approximation:

$$u(\mathbf{x}) = \frac{ikIAe^{ik(r'+s')}}{4\pi r' s'} e^{i\mathbf{k}_2 \cdot \mathbf{r}_1} \int_A \tau(\mathbf{r}) e^{i\mathbf{q} \cdot \boldsymbol{\rho}} e^{ik \left[ \frac{\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2}{2r'} + \frac{(\mathbf{r}_1 - \boldsymbol{\rho})^2 - [\hat{\mathbf{s}}' \cdot (\mathbf{r}_1 - \boldsymbol{\rho})]^2}{2s'} \right]} dS$$

## DIFFRACTION THEORY (4)

- Diffraction pattern of a point source, part 2:

- ▷ Diffraction integral for a point source in Fresnel's approximation:

$$u(\mathbf{x}) = \frac{ikIAe^{ik(r'+s')}}{4\pi r's'} e^{i\mathbf{k}_2 \cdot \mathbf{r}_1} \int_A \tau(\boldsymbol{\rho}) e^{i\mathbf{q} \cdot \boldsymbol{\rho}} e^{ik \left[ \frac{\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2}{2r'} + \frac{(\mathbf{r}_1 - \boldsymbol{\rho})^2 - [\hat{\mathbf{S}}' \cdot (\mathbf{r}_1 - \boldsymbol{\rho})]^2}{2s'} \right]} dS$$

where  $\mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2$ ,  $\mathbf{k}_1 := k\hat{\mathbf{r}}'$ ,  $\mathbf{k}_2 := k\hat{\mathbf{S}}'$

- ▷  $e^{i\mathbf{k}_2 \cdot \mathbf{r}_1}$  = plane wave “carrier” incident at P

- ▷ Integral over aperture = “signal”

- $e^{ik \left[ \frac{\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2}{2r'} + \frac{(\mathbf{r}_1 - \boldsymbol{\rho})^2 - [\hat{\mathbf{S}}' \cdot (\mathbf{r}_1 - \boldsymbol{\rho})]^2}{2s'} \right]}$  = wavefront curvature correction

- **Fresnel diffraction:** Wavefront curvature is an essential part of the physics

- **Fraunhofer diffraction:** No wavefront curvature correction needed

$$u(\mathbf{x}) = \int_A \tau(\boldsymbol{\rho}) e^{i\mathbf{q} \cdot \boldsymbol{\rho}} dS$$

**= Fourier transform of  $\tau$**

## DIFFRACTION THEORY (5)

- Fraunhofer diffraction at a rectangular aperture:

$$u(\mathbf{x}) = \int_A \tau(\boldsymbol{\rho}) e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS$$

where  $\mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2$ ,  $\mathbf{k}_1 := k\hat{\mathbf{r}}'$ ,  $\mathbf{k}_2 := k\hat{\mathbf{s}}'$

- ▷ Aperture dimensions:  $2a$  (in  $x$ )  $\times$   $2b$  (in  $y$ )
- ▷  $\mathbf{q}$  is not necessarily in the plane of the aperture, but we can assume that it is for this case because its  $z$ -component is small
- ▷ Integral over the components of  $\boldsymbol{\rho}$ :

$$\begin{aligned} u(\mathbf{r}_1) &= \frac{ikIAe^{ik(r'+s')}}{4\pi r's'} e^{i\mathbf{k}_2\cdot\mathbf{r}_1} \int_{-a}^a \int_{-b}^b e^{i(q_x\xi + q_y\eta)} d\xi d\eta \\ &= (\text{constant}) [2 \operatorname{sinc}(q_x a)] [2 \operatorname{sinc}(q_y b)] \end{aligned}$$

- ▷ Intensity:

$$S(\mathbf{q}) = S(\mathbf{0}) \operatorname{sinc}^2(q_x a) \operatorname{sinc}^2(q_y b)$$

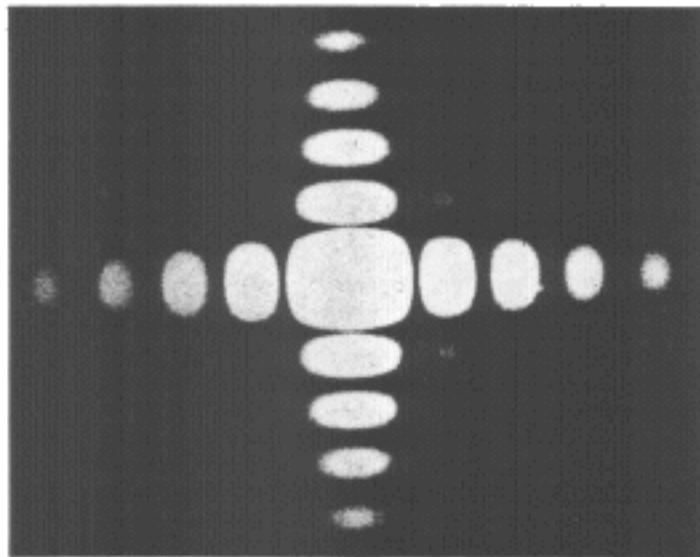


Fig. 8.10. FRAUNHOFER diffraction pattern of a rectangular aperture  $8 \text{ mm} \times 7 \text{ mm}$ , magnification  $50\times$ , mercury yellow light  $\lambda = 5790 \text{ \AA}$ . To show the existence of the weak secondary maxima the central portion was overexposed.

(After H. LIPSON, C. A. TAYLOR, and B. J. THOMPSON.)

## DIFFRACTION THEORY (6)

- Fraunhofer diffraction at a circular aperture:

$$u(\mathbf{x}) = \int_A \tau(\boldsymbol{\rho}) e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS$$

where  $\mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2$ ,  $\mathbf{k}_1 := k\hat{\mathbf{r}}'$ ,  $\mathbf{k}_2 := k\hat{\mathbf{s}}'$

▷ Aperture radius =  $a$

▷ Integral over the components of  $\boldsymbol{\rho}$ :

$$\begin{aligned} u(\mathbf{q}) &= \frac{ikIAe^{ik(r'+s')}}{4\pi r's'} e^{i\mathbf{k}_2\cdot\mathbf{r}_1} \int_0^{2\pi} \int_0^a e^{iq\rho\cos\phi} \rho d\rho d\phi \\ &= (\text{constant})(\pi a^2) \left[ \frac{2J_1(qa)}{qa} \right] \end{aligned}$$

▷ Intensity:

$$S(\mathbf{q}) = S(\mathbf{0}) \left[ \frac{2J_1(qa)}{qa} \right]^2$$

(the **Airy pattern**)

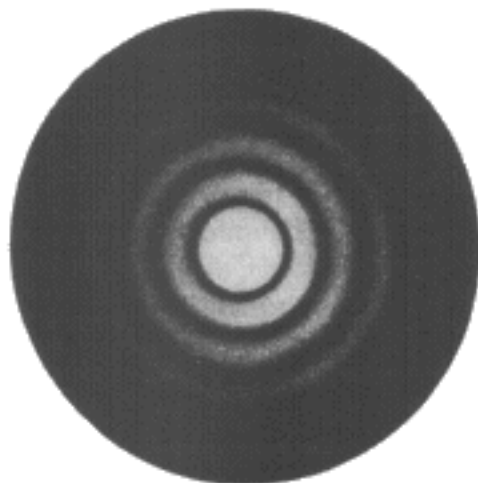


Fig. 8.12. FRAUNHOFER diffraction pattern of a circular aperture (the AIRY pattern) 6 mm in diameter, magnification  $50\times$ , mercury yellow light  $\lambda = 5790 \text{ \AA}$ . To show the existence of the weak subsidiary maxima, the central portion was overexposed.

(After H. LIPSON, C. A. TAYLOR, and B. J. THOMPSON.)

## DIFFRACTION THEORY (7)

- Dimensions of central feature:

- ▷ Rectangular aperture,  $2a \times 2b$

- For a small diffraction angle,  $q_x \approx k \frac{\xi}{s'}$ ,  $q_y \approx k \frac{\eta}{s'}$
    - First zero of sinc function is at  $x = \pi$
    - At the first zero in  $\xi$ ,  $\frac{2\pi \xi_0}{\lambda} \frac{a}{s'} = \pi$

$$\xi_0 = \frac{\lambda s'}{2a}, \quad \eta_0 = \frac{\lambda s'}{2b}$$

- ▷ Circular aperture, radius  $a$

- For a small diffraction angle  $\theta$ ,  $q \approx k\theta$
    - First zero of  $J_1$  is  $j_{1,1} = 3.832 \dots = 1.220\pi$
    - At the first zero,  $qa \approx 2\pi\theta_0 a / \lambda \approx 1.220\pi$

$$\theta_0 \approx 0.610 \frac{\lambda}{a} \approx 1.220 \frac{\lambda}{d}$$

## DIFFRACTION THEORY (8)

- Fraunhofer diffraction by  $N$  randomly arranged identical apertures:
  - ▷ Aperture  $A_m$  is displaced from aperture  $A_1$  by a random vector  $\mathbf{a}_m$
  - ▷ Amplitude of diffracted field is proportional to

$$\int_A e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS = \sum_{m=1}^N e^{i\mathbf{q}\cdot\mathbf{a}_m} \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS$$

- ▷ Intensity of diffracted field is

$$\begin{aligned} S(\mathbf{q}) &= (\text{const.}) \left| \sum_{m=1}^N e^{i\mathbf{q}\cdot\mathbf{a}_m} \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS \right|^2 \\ &= (\text{const.}) \sum_{m=1}^N \sum_{n=1}^N e^{i\mathbf{q}\cdot(\mathbf{a}_m - \mathbf{a}_n)} \left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS \right|^2 = N \times (\text{const.}) \times \left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS \right|^2 \\ &\quad \left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS \right|^2 \propto \text{diffraction pattern of a single aperture} \end{aligned}$$

- ▷ Intensities add, not amplitudes, when the arrangement is random

## DIFFRACTION THEORY (9)

- Fraunhofer diffraction by periodically arranged identical apertures, part 1:
  - ▷ The same unit cell is translated through integer multiples of the vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$  (not necessarily orthogonal!)
  - ▷ Amplitude of diffracted field is proportional to

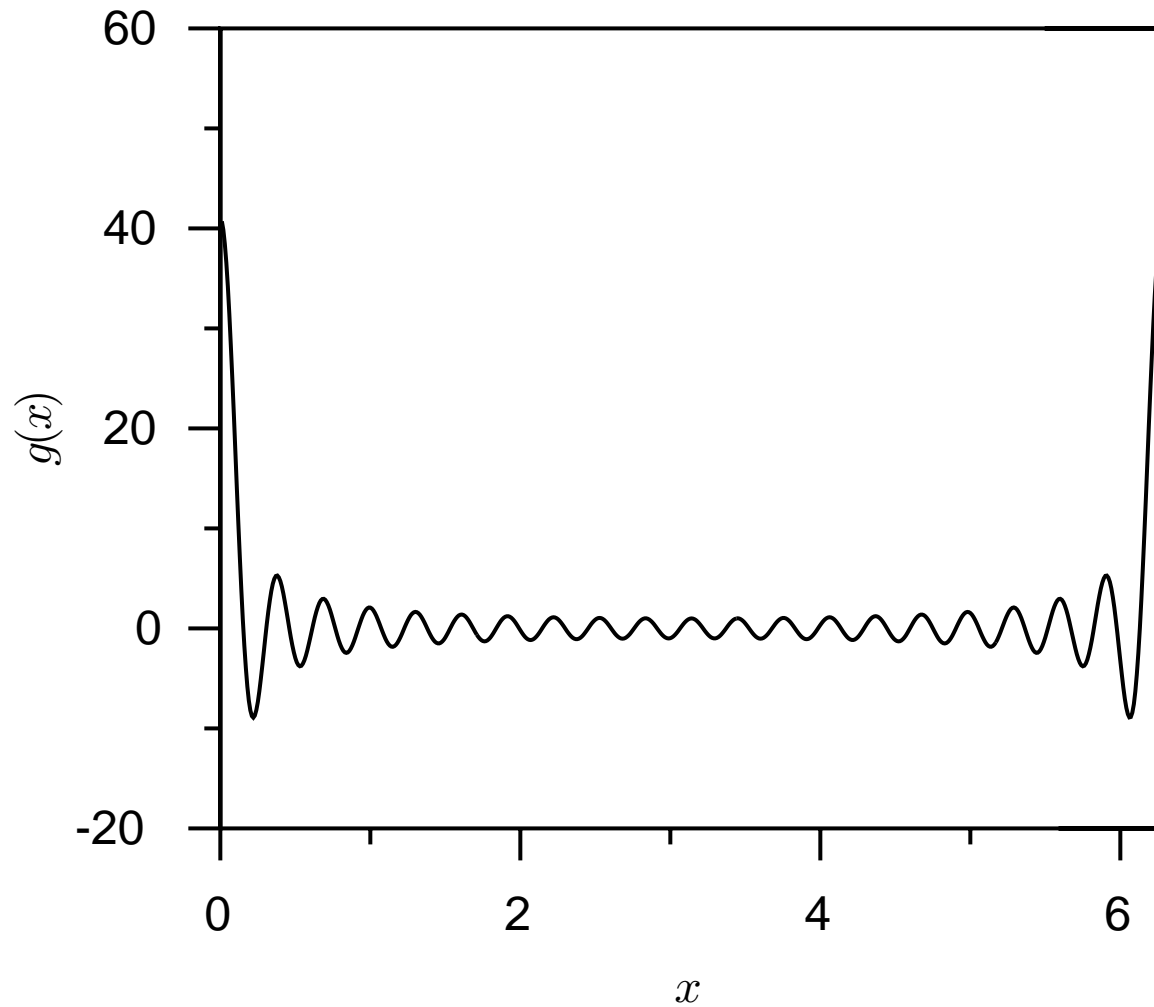
$$\begin{aligned} \int_A e^{i\mathbf{q}\cdot\rho} dS &= \sum_{m_1=-M_1}^{M_1} \sum_{m_2=-M_2}^{M_2} e^{i\mathbf{q}\cdot(m_1\mathbf{a}_1+m_2\mathbf{a}_2)} \int_{A_1} e^{i\mathbf{q}\cdot\rho} dS \\ &= g_1(\mathbf{q}\cdot\mathbf{a}_1)g_2(\mathbf{q}\cdot\mathbf{a}_2) \int_{A_1} e^{i\mathbf{q}\cdot\rho} dS \end{aligned}$$

- ▷ Grating functions:

$$\begin{aligned} g_j(\mathbf{q}\cdot\mathbf{a}_j) &= \sum_{m_j=-M_j}^{M_j} e^{im_j\mathbf{q}\cdot\mathbf{a}_j} \\ &= \frac{\sin\left[\left(M_j + \frac{1}{2}\right)\mathbf{q}\cdot\mathbf{a}_j\right]}{\sin\frac{1}{2}\mathbf{q}\cdot\mathbf{a}_j} \end{aligned}$$

# THE GRATING FUNCTION

$$g(x) = \frac{\sin[(M + \frac{1}{2})x]}{\sin(\frac{1}{2}x)} \text{ for } M = 20$$



## DIFFRACTION THEORY (10)

- Fraunhofer diffraction by periodically arranged identical apertures, part 2:

▷ Properties of the grating function

$$g(\mathbf{q} \cdot \mathbf{a}) = \frac{\sin \left[ \left( M + \frac{1}{2} \right) \mathbf{q} \cdot \mathbf{a} \right]}{\sin \frac{1}{2} \mathbf{q} \cdot \mathbf{a}}$$

○ Peak value of  $g$  is  $2M + 1$  at

$$\mathbf{q} \cdot \mathbf{a} = 2n\pi$$

○ Half-width of central peak =  $\frac{\pi}{M + \frac{1}{2}}$

○ Amplitude of first side lobe is  $-0.212 \times$  amplitude of central peak

## DIFFRACTION THEORY (11)

- Fraunhofer diffraction by periodically arranged identical apertures, part 3:

▷ Intensity of diffracted field is proportional to

$$\left| \int_A e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS \right|^2 = |g_1(\mathbf{q}\cdot\mathbf{a}_1)|^2 |g_2(\mathbf{q}\cdot\mathbf{a}_2)|^2 \left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS \right|^2$$

- The grating functions are sharply peaked at values of  $\mathbf{q}$  such that

$$\mathbf{q}\cdot\mathbf{a}_1 = 2n_1\pi \quad \text{and} \quad \mathbf{q}\cdot\mathbf{a}_2 = 2n_2\pi$$

- ◊ These conditions imply that  $\mathbf{q}$  is  $2\pi$  times a reciprocal lattice vector:

$$\mathbf{q} = 2\pi(n_1\mathbf{b}_1 + n_2\mathbf{b}_2)$$

- Peaks are modulated by the diffraction pattern of a single aperture,

$$\left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}} dS \right|^2$$

## 2-D RECIPROCAL LATTICE (1)

- The 2-d direct lattice spanned by non-parallel vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$  consists of the points

$$\mathbf{r}_{m_1, m_2} = m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2$$

- The **reciprocal lattice** to the  $\mathbf{a}_1 - \mathbf{a}_2$  lattice consists of vectors

$$\mathbf{k}_{n_1, n_2} = n_1 \mathbf{b}_1 + n_2 \mathbf{b}_2$$

where

$$\mathbf{b}_i \cdot \mathbf{a}_j = \delta_{i,j}$$

▷ The **reciprocal lattice basis** is

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \hat{\mathbf{z}}}{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}, \quad \mathbf{b}_2 = \frac{\hat{\mathbf{z}} \times \mathbf{a}_1}{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}$$

Hexagonal lattice basis:  $\mathbf{a}_1 = \hat{\mathbf{x}}, \quad \mathbf{a}_2 = \frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}}$

Reciprocal lattice basis:  $\mathbf{b}_1 = \hat{\mathbf{x}} - \frac{1}{\sqrt{3}}\hat{\mathbf{y}}, \quad \mathbf{b}_2 = \frac{2}{\sqrt{3}}\hat{\mathbf{y}}$

**DIFFRACTION THEORY (12)**

- Calculation of the field in the image plane, part 1:

- ▷ Assume normal incidence

- Inclination factor

$$I = -(\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}}) = -2$$

- Diffraction integral assuming a point source and a thin lens with focal length  $f$ :

$$\begin{aligned} u(\mathbf{r}_1) &= -\frac{iA}{\lambda} \int_A \frac{e^{ik(r+s)}}{rs} \tau(\boldsymbol{\rho}) dS \\ &\approx -\frac{iA}{\lambda} \int_A \frac{e^{ik(r+s)}}{rs} e^{-ik\frac{\rho^2}{2f}} \alpha(\boldsymbol{\rho}) d^2\rho \end{aligned}$$

- ◊  $\tau$  = transmission function of lens and aperture

- ◊  $\alpha$  = transmission function of aperture alone

## DIFFRACTION THEORY (13)

- Calculation of the field in the image plane, part 2:
  - ▷ Diffraction integral assuming a point source and a thin lens with focal length  $f$ :

$$u(\mathbf{r}_1) \approx -\frac{iA}{\lambda} \int_A \frac{e^{ik(r+s)}}{rs} e^{-ik\frac{\rho^2}{2f}} \alpha(\boldsymbol{\rho}) d^2\rho$$

- The sum of the exponents is (in the Fresnel approximation)

$$\begin{aligned} ik \left[ r + s - \frac{\rho^2}{2f} \right] &\approx ik \left[ r' + s' - \frac{\rho^2}{2f} + \frac{(\mathbf{r}_0 - \boldsymbol{\rho})^2}{2r'} + \frac{(\boldsymbol{\rho} - \mathbf{r}_1)^2}{2s'} \right] \\ &= ik \left[ r' + s' + \frac{\rho^2}{2} \left( -\frac{1}{f} + \frac{1}{r'} + \frac{1}{s'} \right) + \frac{r_0^2}{2r'} + \frac{r_1^2}{2s'} - \boldsymbol{\rho} \cdot \left( \frac{\mathbf{r}_0}{r'} + \frac{\mathbf{r}_1}{s'} \right) \right] \end{aligned}$$

## DIFFRACTION THEORY (14)

- Calculation of the field in the image plane, part 3:

▷ At the Gaussian focal point of a thin lens,

$$-\frac{1}{f} + \frac{1}{r'} + \frac{1}{s'} = 0$$

- This makes the coefficient of  $\rho^2$  vanish in the exponent
- The diffraction integral reduces to a phase factor times the Fourier transform of the **pupil function**  $\alpha$ :

$$u(\mathbf{r}_1) \approx -\frac{iA}{\lambda r' s'} e^{ik \left[ \left( \frac{r_0^2}{2r'} + \frac{r_1^2}{2s'} \right) + r' + s' \right]} \int_A \alpha(\boldsymbol{\rho}) e^{-ik\boldsymbol{\rho} \cdot \left( \frac{\mathbf{r}_0}{r'} + \frac{\mathbf{r}_1}{s'} \right)} d^2\rho$$

- The image is spread out by diffraction
- The peak amplitude occurs at the stationary-phase point:

$$\frac{\mathbf{r}_0}{r'} + \frac{\mathbf{r}_1}{s'} = \mathbf{0} \quad \Rightarrow \quad \mathbf{r}_1 = -\frac{s'}{r'} \mathbf{r}_0 \quad (\text{Gaussian image displacement})$$

## DIFFRACTION THEORY (15)

- Imaging by a thin lens of an extended, coherently illuminated object:
  - ▷ Superpose the fields due to a distribution  $\psi$  of point sources:

$$u(\mathbf{r}_1) \approx -\frac{i}{\lambda} \int_S \int_A \psi(\mathbf{r}_0) \frac{e^{ikr}}{r} \alpha(\boldsymbol{\rho}) e^{-ik\frac{\rho^2}{2f}} \frac{e^{iks}}{s} d^2\rho d^2r_0$$

- Instead of performing two 2-d integrations simultaneously, we can evaluate a diffraction integral to go from the source to the lens and another to go from the lens to the observation plane
- Field incident at lens:

$$\begin{aligned} u_{\text{inc}}(\boldsymbol{\rho}) &\approx -\frac{i}{\lambda} \frac{e^{ikr'}}{r'} \int_S \psi(\mathbf{r}_0) e^{\frac{ik(\boldsymbol{\rho}-\mathbf{r}_0)^2}{2r'}} d^2r_0 \\ &\approx -\frac{i}{\lambda} \frac{e^{ikr'}}{r'} \int_S \psi(\mathbf{r}_0) F_{k/r'}(\boldsymbol{\rho} - \mathbf{r}_0) d^2r_0 \end{aligned}$$

- ◊  $u_{\text{inc}}$  is a convolution of the source distribution  $\psi$  with the Fresnel function  $F_{k/r'}$ , where

$$F_\alpha(\mathbf{r}) := e^{i\alpha r^2}$$

**THE FRESNEL FUNCTION (1)**

- The **Fresnel function** is

$$F_\alpha(\mathbf{r}) := e^{\frac{1}{2}i\alpha r^2}$$

where  $\mathbf{r}$  lies in the  $x - y$  plane

- ▷ The 2-d Fourier transform of  $F_\alpha$  is

$$\begin{aligned}\tilde{F}_\alpha(\mathbf{q}) &= \int e^{-i\mathbf{q}\cdot\mathbf{r}} F_\alpha(\mathbf{r}) d^2r \\ &= \frac{2i\pi}{\alpha} F_{-1/\alpha}(\mathbf{q}) \\ &= \frac{2i\pi}{\alpha} e^{-i\frac{q^2}{2\alpha}}\end{aligned}$$

## DIFFRACTION THEORY (16)

- Imaging by a thin lens of an extended, coherently illuminated object, part 2:
  - ▷ The field incident on the lens,  $u_{\text{inc}}$ , is a convolution of the source distribution  $\psi$  with the Fresnel function  $F_{k/r'}$ :

$$\begin{aligned} u_{\text{inc}}(\boldsymbol{\rho}) &= -\frac{i}{\lambda} \frac{e^{ikr'}}{r'} \int_S \psi(\mathbf{r}_0) F_{k/r'}(\boldsymbol{\rho} - \mathbf{r}_0) d^2r_0 \\ &= -\frac{i}{\lambda} \frac{e^{ikr'}}{r'} \frac{1}{(2\pi)^2} \int \tilde{\psi}(\mathbf{q}) \tilde{F}_{k/r'}(\mathbf{q}) e^{i\mathbf{q}\cdot\boldsymbol{\rho}} d^2q \end{aligned}$$

- ▷ Diffraction integral for the field in the observation plane:

$$u(\mathbf{r}_1) = \left(-\frac{i}{\lambda}\right)^2 \frac{e^{ik(r'+s')}}{(2\pi)^2 r' s'} \int \int_A \tilde{\psi}(\mathbf{q}) \tilde{F}_{k/r'}(\mathbf{q}) \alpha(\boldsymbol{\rho}) e^{E(\mathbf{q}, \boldsymbol{\rho}, \mathbf{r}_1)} d^2\rho d^2q$$

- The sum of the exponents is (in the Fresnel approximation)

$$E(\mathbf{q}, \boldsymbol{\rho}, \mathbf{r}_1) = ik \left[ -\frac{\rho^2}{2f} + \frac{\rho^2}{2s'} + \frac{r_1^2}{2s'} \right] + i \left( \mathbf{q} - k \frac{\mathbf{r}_1}{s'} \right) \cdot \boldsymbol{\rho}$$

## DIFFRACTION THEORY (17)

- Imaging by a thin lens of an extended, coherently illuminated object, part 3:

▷ In the back focal plane

$$s' = f$$

the sum of the exponents does not depend on  $\rho^2$ :

$$E(\mathbf{q}, \rho, \mathbf{r}_1) = ik \frac{r_1^2}{2s'} + i\mathbf{q}' \cdot \rho$$

where

$$\mathbf{r}'_1 := \mathbf{r}_1|_{s'=f} \quad \text{and} \quad \mathbf{q}' := \mathbf{q} - k \frac{\mathbf{r}'_1}{s'}$$

▷ Diffraction integral for the field in the back focal plane:

$$u(\mathbf{r}'_1) = \left(-\frac{i}{\lambda}\right)^2 \frac{e^{ik(r'+s')}}{(2\pi)^2 r' s'} e^{ik \frac{r_1'^2}{2f}} \int \tilde{\psi}(\mathbf{q}) \tilde{F}_{k/r'}(\mathbf{q}) \tilde{\alpha}(\mathbf{q}') d^2q$$

## DIFFRACTION THEORY (18)

- Imaging by a thin lens of an extended, coherently illuminated object, part 4:

▷ Diffraction integral for the field in the back focal plane:

$$u(\mathbf{r}'_1) = \left(-\frac{i}{\lambda}\right)^2 \frac{e^{ik(r'+s')}}{(2\pi)^2 r' s'} e^{ik\frac{r'^2_1}{2f}} \int \tilde{\psi}(\mathbf{q}) \tilde{F}_{k/r'}(\mathbf{q}) \tilde{\alpha}(\mathbf{q}') d^2q$$

▷ The Fourier transform of the pupil function,  $\tilde{\alpha}$ , is very sharply peaked at  $\mathbf{q}' = \mathbf{0}$  (for a circular pupil,  $\tilde{\alpha}$  is the Airy disk)

- For an infinite aperture,  $\tilde{\alpha}(\mathbf{q}') = (2\pi)^2 \delta(\mathbf{q}') \Rightarrow \mathbf{q} = k \frac{\mathbf{r}'_1}{f}$
- In this limit, and with the Fraunhofer approximation,

$$u(\mathbf{r}'_1) \approx -\frac{i}{\lambda f} \tilde{\psi} \left( k \frac{\mathbf{r}'_1}{f} \right)$$

- **The back focal plane contains the Fourier transform of a coherently illuminated object**

## DIFFRACTION THEORY (19)

- Imaging by a thin lens of an extended, coherently illuminated object, part 5: **Focal plane filtering**
  - ▷ Concept: Alter the image by (intentionally or unintentionally) filtering the Fourier transform in the focal plane
  - ▷ Propagate the field from the back focal plane to the image plane using yet another diffraction integral!
  - ▷ Field in the image plane:

$$u(\mathbf{r}_1) = -\frac{i}{\lambda} \int_{\text{F}} \frac{e^{iku}}{u} \sigma(\mathbf{r}'_1) u(\mathbf{r}'_1) d^2r'_1$$

- Focal-plane filter transmission function =  $\sigma$
- Vector from point in focal plane to point in image plane:

$$\mathbf{u} = \mathbf{s}'' + \mathbf{r}_1 - \mathbf{r}'_1$$

## DIFFRACTION THEORY (20)

- Imaging by a thin lens of an extended, coherently illuminated object, part 6
  - ▷ Propagate the field from the back focal plane to the image plane
    - Assume an infinite aperture,  $\tilde{\alpha}(\mathbf{q}') = (2\pi)^2 \delta(\mathbf{q}')$ :

$$u(\mathbf{r}'_1) \approx -\frac{i}{\lambda f} e^{\frac{ikr'_1{}^2}{2} \left( \frac{1}{f} - \frac{r'}{f^2} \right)} \tilde{\psi} \left( k \frac{\mathbf{r}'_1}{s'} \right)$$

- Sum of exponents in the diffraction integral:

$$\frac{ikr'_1{}^2}{2} \left( \frac{1}{f} - \frac{r'}{f^2} + \frac{1}{s''} \right)$$

- Use Newton's equation for the focus-to-image distance  $r_I$ :

$$\frac{1}{f} - \frac{r'}{f^2} = \frac{f - r'}{f^2} = \frac{1}{f - r_I}$$

- All quadratic phase terms cancel in the image plane,  $f - r_I = -s''$

## DIFFRACTION THEORY (21)

- Imaging by a thin lens of an extended, coherently illuminated object, part 7
  - ▷ Field amplitude in the image plane:

$$u(\mathbf{r}'_1) \approx - \left( \frac{i}{\lambda} \right)^2 \frac{1}{(2\pi)^2 s'' f} e^{i \frac{k r_1^2}{2s''}} \int_{\mathbb{F}} \sigma(\mathbf{r}'_1) \tilde{\psi} \left( k \frac{\mathbf{r}'_1}{f} \right) e^{-i k \frac{\mathbf{r}'_1 \cdot \mathbf{r}_1}{s''}} d^2 r'_1$$

- The image-plane field is proportional to the Fourier transform of the product of the focal-plane filtering function  $\sigma$  and the Fourier transform of the original field