Chapter 16

Partial differential equations

16.1 Introduction

Partial differential equations (PDEs) play a major role in electromagnetic engineering and mechanical engineering, as well as most fields of physics. Methods for solving PDEs depend on the order of the PDE (the order of the highest partial derivative) and, for PDEs of order 2 or greater, on the type of the PDE. We discuss first-order PDEs, second-order PDEs, and then a few essential aspects of numerical methods.

An example of a second-order PDE that is of considerable importance in modern communications technology is the beam-propagation equation

$$\frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} + \frac{i}{2}\beta_2\frac{\partial^2 u}{\partial t^2} = -\frac{\alpha}{2}u + i\gamma|u|^2 u, \tag{16.1}$$

which describes the propagation of a guided optical wave with an electric-field intensity that is proportional to u along a fiber that extends in the z direction. In this equation, the term $-\alpha u/2$ describes linear attenuation, the term $\beta_2\partial^2 u/\partial t^2$ describes chromatic dispersion, which broadens pulses in time, and the term $\gamma|u|^2 u$ describes some of the nonlinear-optical properties of the fiber such as self-phase modulation, cross-phase modulation and four-wave mixing. The design of a fiber transmission system typically involves many numerical simulations using the beam-propagation equation.

An ordinary differential equation such as

$$\frac{dz}{dt} = f(z,t) \tag{16.2}$$

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determines a family of curves $z(t)$, from which the initial condition or a boundary condition selects one curve as the solution. However, a partial differential equation in two independent variables z, t determines a family of surfaces $u(z,t)$, from which an initial condition, a set of boundary conditions, or a combination of initial and boundary conditions selects one surface as the solution.

There are three different types of second-order PDEs, called hyperbolic, parabolic and elliptic. PDEs of hyperbolic type describe initial- and boundary-value problems involving wave motion; PDEs of parabolic type describe either initial-value diffusion processes or one-way wave motion; and PDEs of elliptic type describe boundary-value problems such as occur in electrostatics. Each type of second-order PDE requires its own set of numerical methods. Methods for hyperbolic equations will not work for elliptic equations, for example.

16.2 First-order, quasilinear PDEs

16.2.1 Basic definitions

A first-order partial differential equation of the form

$$\mathbf{a}(u,\mathbf{r}) \cdot \nabla u(\mathbf{r}) = c(\mathbf{r}) \tag{16.3}$$

(where $\mathbf{r} \in \mathbb{R}^n$ and u is the unknown function to be solved for) is called **quasilinear**. If the source function c is the null function, $c \equiv 0$, then the equation is called **homogeneous**. If c is not the null function, then the equation is **inhomogeneous**. Methods for the homogeneous and inhomogeneous cases differ in some essential respects.

We shall use the beam-propagation equation with dispersion and linear attenuation turned off,

$$\frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} = i\gamma |u|^2 u, \tag{16.4}$$

to illustrate both analytical and numerical methods for first-order quasilinear PDEs. In this example,

$$a = 1, \quad b = \frac{1}{v}, \quad c = i\gamma|u|^2 u. \tag{16.5}$$

If we also set $\gamma = 0$, then the beam-propagation is homogeneous; otherwise, it is inhomogeneous.

The solutions of a homogeneous quasilinear partial differential equation do not obey the superposition principle if the coefficient vector $\mathbf{a}$ depends non-trivially on the solution u. For example, if u_1 and u_2 are both solutions of **Burgers' equation**,

$$u\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = 0, \tag{16.6}$$

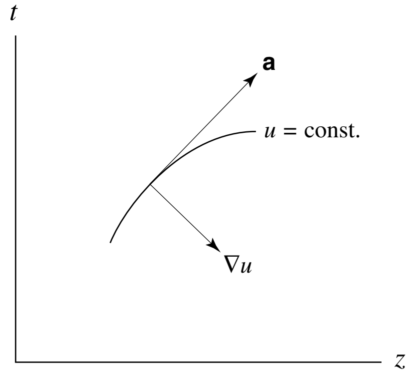

Figure 16.1: The vector $\mathbf{a}(u, \mathbf{r})$ is orthogonal to ∇u, and is tangent to the curve $u =$ constant.

then αu_1 is not a solution unless $\alpha = 1$, because (unless $u_1 \equiv 0$)

$$\alpha^2 u_1 \frac{\partial u_1}{\partial z} + \alpha \frac{\partial u_1}{\partial t} = (\alpha^2 - 1) u_1 \frac{\partial u_1}{\partial z} + (\alpha - 1) \frac{\partial u_1}{\partial t} \neq 0, \tag{16.7}$$

and $u_1 + u_2$ is not a solution unless $u_1 \equiv 0$ or $u_2 \equiv 0$ (or both), because

$$(u_1 + u_2) \frac{\partial (u_1 + u_2)}{\partial z} + \frac{\partial (u_1 + u_2)}{\partial t} = u_1 \frac{\partial u_2}{\partial z} + u_2 \frac{\partial u_1}{\partial z} \neq 0. \tag{16.8}$$

The inapplicability of the superposition principle makes the theory of quasilinear PDEs less than straightforward, because one cannot appeal for help from linear algebra.

16.2.2 First-order, quasilinear, homogeneous PDEs

16.2.2.1 Introduction and motivation

The homogeneous equation

$$\mathbf{a}(u, \mathbf{r}) \cdot \nabla u(\mathbf{r}) = 0 \tag{16.9}$$

implies that the vector $\mathbf{a}(u, \mathbf{r})$ is orthogonal to ∇u at every point $\mathbf{r}$ of the domain of u (Fig. 16.1). Since ∇u is orthogonal to the level surfaces of u, the vector $\mathbf{a}$ is tangent to the level surface

$$u(\mathbf{r}) = \text{constant} \tag{16.10}$$

at every $\mathbf{r} \in$ domain $[u]$. Thus the equation $\mathbf{a} \cdot \nabla u = 0$ defines the level surfaces of u by defining a family of tangent lines or planes. We show how to construct the function u by constructing its level surfaces.

16.2.2.2 Characteristics in the two-dimensional case

We specialize now to the two-dimensional case, in which Eq. (16.9) reads

$$a(u,z,t)\frac{\partial u}{\partial z}+b(u,z,t)\frac{\partial u}{\partial t}=0 \tag{16.11}$$

where

$$z=x^1, \quad t=x^2, \quad a=a^1, \quad b=a^2. \tag{16.12}$$

This equation implies that the vector field

$$\mathbf{a}(u,z,t)=\begin{pmatrix} a(u,z,t) \\ b(u,z,t) \end{pmatrix} \tag{16.13}$$

is tangent to the curve in the x–t plane that is defined by the condition $u(z,t) =$ constant. It is equivalent to say that the derivative of u in a direction that is parallel to $\mathbf{a}$ is zero, because the directional derivative of u in the direction that is specified by the unit vector $\hat{\mathbf{s}}$ is $\hat{\mathbf{s}} \cdot \nabla u$.

For example, Burgers' equation (16.6) is a homogeneous quasilinear equation of the form (16.11) with $a(u,z,t) = u$ and $b(u,z,t) = 1$. The left-hand side of Burgers' equation is a special case of the convective derivative of the velocity, $\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v}$, which occurs in the Navier-Stokes equations of fluid dynamics.

Let C be a rectifiable curve in the z–t plane, and let

$$z=z(s), \quad t=t(s) \tag{16.14}$$

be the parametric equations of C in terms of the arc length s. The derivative of u along C is

$$\frac{du}{ds}=\frac{\partial u}{\partial z}\frac{dz}{ds}+\frac{\partial u}{\partial t}\frac{dt}{ds}. \tag{16.15}$$

If C is a curve on which the function u has a constant value, then $du/ds = 0$ on C. Then Eq. (16.15) implies that the vector

$$\mathbf{t}(u,z,t)=\begin{pmatrix} \dfrac{dz}{ds} \\ \dfrac{dt}{ds} \end{pmatrix} \tag{16.16}$$

is tangent to C.

We now have two vector fields, $\mathbf{a}$ $\big($Eq. (16.13)$\big)$ and $\mathbf{t}$ $\big($Eq. (16.16)$\big)$, both of which are tangent to the curve C, on which u has a constant value. It follows that $\mathbf{a}$ and $\mathbf{t}$ are parallel. Therefore, for each value of arc length (s), $\mathbf{t}$ is proportional to $\mathbf{a}$:

$$\mathbf{t}(s)=F(s)\,\mathbf{a}(s). \tag{16.17}$$

It follows that

$$\begin{aligned} \frac{dz}{ds} &= F(s)\, a\Big(u\big(z(s), t(s)\big), z(s), t(s)\Big), \\ \frac{dt}{ds} &= F(s)\, b\Big(u\big(z(s), t(s)\big), z(s), t(s)\Big). \end{aligned} \tag{16.18}$$

For a rectifiable curve in $\mathbb{R}^2$, the tangent vector $\mathbf{t}$ is a unit vector:

$$\mathbf{t} \cdot \mathbf{t} = \left(\frac{dz}{ds}\right)^2 + \left(\frac{dt}{ds}\right)^2 = 1. \tag{16.19}$$

It follows that dz/ds and dt/ds cannot vanish simultaneously. As long as a and b are finite and nonzero, it follows that $F(s) \neq 0$. The next step is to show how to use Eq. (16.18) to determine the curve C and the solution, u.

Dividing the first equation in (16.18) by the second equation, one obtains the ordinary differential equation

$$\boxed{\frac{dz}{dt} = \frac{dz/ds}{dt/ds} = \frac{a}{b}} \tag{16.20}$$

which determines the curve C. A curve determined by this differential equation is called a **characteristic curve**.

16.2.2.3 The PDE along a characteristic curve

From Eqs. (16.15) and (16.18) one has

$$\frac{du}{ds} = F(s)\left(a\frac{\partial u}{\partial z} + b\frac{\partial u}{\partial t}\right). \tag{16.21}$$

From this equation and the original partial differential equation (16.11) one sees that, on a characteristic curve,

$$\boxed{\frac{du}{ds} = F(s)\, c\Big(u\big(z(s), t(s)\big), z(s), t(s)\Big).} \tag{16.22}$$

In the homogeneous case ($c = 0$), u is constant along a characteristic curve $\big(du/ds = 0\big)$.

Eq. (16.22), which describes the rate of change of u as one goes along the characteristic curve, replaces the original PDE, Eq. (16.11). Since differentiation with respect to s may not be convenient, we re-express du/ds in terms of a derivative with respect to z along the characteristic curve, in order to have a more useful expression than (16.22) for the inhomogeneous case. From the relation $F = a^{-1}dz/ds$, Eq. (16.66), one has

$$\frac{1}{F}\frac{du}{ds} = a\left(\frac{du}{dz}\right)_C \tag{16.23}$$

where $(du/dz)_C$ is the rate of change of u as z is varied along the characteristic curve C. For regions in which a does not vanish, one can therefore replace the original PDE or Eq. (16.22) with the equation

$$\boxed{\left(\frac{du}{dz}\right)_C = \frac{\partial u}{\partial z} + \frac{b}{a}\frac{\partial u}{\partial t} = \frac{c}{a}.} \tag{16.24}$$

The subscript C indicates that the derivative with respect to z is evaluated along the characteristic curve C, according to Eq. (16.23). Again, in the homogeneous case $(c = 0)$, $(du/dz)_C = 0$.

Conversely, let z obey the ordinary differential equation $dz/dt = a/b$ [Eq. (16.20)] and let C be the curve determined by this differential equation. It follows that Eq. (16.21) holds along C. Let s be the arc length along C, and let u be any adequately differentiable function that is constant along C. Then $du/ds = 0$. By the chain rule (16.15),

$$\begin{aligned} \frac{du}{ds} &= \frac{\partial u}{\partial z}\frac{dz}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} \\ &= Fa\frac{\partial u}{\partial z} + Fb\frac{\partial u}{\partial t}. \end{aligned} \tag{16.25}$$

Since $F(s) \neq 0$, this equation and the equation $du/ds = 0$ imply Eq. (16.11). Therefore a function u satisfies the homogeneous equation (16.11) if and only if u is constant along the characteristic curve defined by Eq. (16.20).

It follows that the general solution of the two-dimensional, homogeneous, quasilinear partial differential equation (16.11) is

$$\boxed{\begin{aligned} u &= k = \text{constant} \\ \frac{dz}{dt} &= \frac{a(k,z,t)}{b(k,z,t)}. \end{aligned}} \tag{16.26}$$

The second equation determines a characteristic curve C along which $u = k$.

16.2.2.4 The homogenous beam-propagation equation

The first-order, homogeneous beam-propagation equation (without attenuation) is

$$\frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} = 0. \tag{16.27}$$

From Eq. (16.5), which gives the coefficients a and b, and the general solution Eq. (16.26), the solution of the first-order, homogeneous beam-propagation equation (without attenuation) is

$$\begin{aligned} u &= k = \text{constant} \\ \frac{dz}{dt} &= v. \end{aligned} \tag{16.28}$$

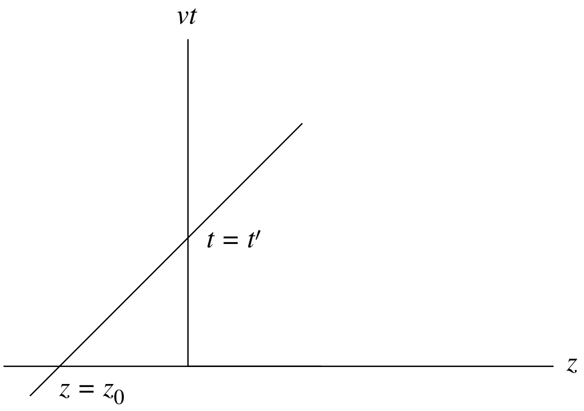

Figure 16.2: The characteristic $z - vt = z_0$ of the beam-propagation equation intercepts the z-axis at z_0 and the t-axis at $t' = -z_0/v$.

Therefore each characteristic curve of the beam-propagation equation is a line

$$z - vt = \text{constant} = z_0. \tag{16.29}$$

This means that the value of u is uniquely determined by the value of z_0, *i.e.*, that

$$u = f(z - vt). \tag{16.30}$$

This equation describes a pulse or wave that moves with a constant velocity v and with a waveform determined by the function f.

In this example, the function F is just a constant. From Eq. (16.66) and Fig. 16.2,

$$ds = \sqrt{2}\, dz \Rightarrow F(s) = \frac{1}{\sqrt{2}}. \tag{16.31}$$

The simplicity of this relation is a result of the fact that the characteristic curves in this example are straight lines.

Regarding u as a function of z_0, which is the z-intercept (the value of z at $t = 0$) for a particular characteristic line amounts to taking a "snapshot" of the wave represented by u at time $t = 0$, and then using the snapshot as an initial condition for u. Usually it is more convenient to regard u as a function of the t-intercept, t' (the value of t for which $z = 0$ for a particular characteristic), because $u(t')$ is the waveform that is "transmitted" from the point $z = 0$. Fig. 16.2 illustrates that, because t' is the t-intercept of the line $z - vt = z_0$, it follows that

$$-vt' = z_0. \tag{16.32}$$

Therefore

$$t' = -\frac{z_0}{v} = -\frac{z - vt}{v} = t - \frac{z}{v}. \tag{16.33}$$

Physically, t' is time referred to a new origin that depends on z. The relation $t' = 0$ defines t as the time at which a short pulse that starts at $z = 0$ at $t = 0$ arrives at z. The equation

$$u = g(t') \tag{16.34}$$

describes the solution of the beam-propagation equation (without attenuation, optical nonlinearities or dispersion) in terms of the time-dependent waveform, g, observed at a particular location, z.

For consistency with the point of view that t' is time measured in a new coordinate system, we define the transformed z coordinate as $z' = z$. The equations

$$\begin{aligned} z' &= z, \\ t' &= t - \frac{z}{v} \end{aligned} \tag{16.35}$$

define the new coordinates in terms of the old. Eq. (16.24), which replaces the original PDE, becomes in this case

$$\boxed{\left(\frac{du}{dz'}\right)_C = \frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} = 0.} \tag{16.36}$$

From Eq. (16.34), the general solution of this equation is

$$u(z,t) = f(z - vt) \tag{16.37}$$

where f is any adequately differentiable function of one variable.

16.2.2.5 Burgers' equation and shock waves

If one regards u as the amplitude of a wave or signal, then the wave has a constant amplitude along C. The slope of the characteristic curve, dz/dt, is equal to the velocity of the wave. For example, the solution of Burgers' equation (16.6) is

$$\begin{aligned} u &= v = \text{constant} \\ \frac{dz}{dt} &= v \end{aligned} \tag{16.38}$$

along a characteristic curve. Therefore each characteristic curve of Burgers' equation is a line of the same form as Eq. (16.29) with $v = u$. The general solution of Burgers' equation is

$$u(z,t) = f(z - ut), \tag{16.39}$$

where f is any once-differentiable, real-valued function on $\mathbb{R}$. This solution defines the function u implicitly, since u occurs on both sides of the equation.

Obviously the general solution (16.26) does not determine u uniquely, because that equation just says that u is constant along a characteristic curve and gives a differential equation that every characteristic curve must satisfy. In order to pick out a particular characteristic curve, one must specify an initial condition or boundary value. Let N be any curve such that the slope of N at any point $\mathbf{r}$ is not equal to the slope of the characteristic curve through $\mathbf{r}$. It can be shown that the values of u on N uniquely determine u, and conversely. In the example of Burgers' equation, one can take N to be the real axis, *i.e.,* the curve defined by the equations $z = s$, $t = 0$. On N,

$$u(z,0) = f(z) \tag{16.40}$$

is the initial value of u. The values of f on N uniquely determine the value of u at every point of every characteristic curve that issues from N.

Let us verify that (16.39) is indeed a solution of Burgers' equation. By the chain rule, one has

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial f}{\partial z} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} \\ &= f' - f't\frac{\partial u}{\partial z}.\end{aligned} \tag{16.41}$$

Collecting terms, one gets the result that

$$\frac{\partial u}{\partial z} = \frac{f'}{1+f't}, \tag{16.42}$$

where f' is the derivative of f with respect to its argument. One shows similarly that

$$\frac{\partial u}{\partial t} = -\frac{f'u}{1+f't}. \tag{16.43}$$

It follows immediately that $u\partial u/\partial z + \partial u/\partial t = 0$.

We show now that u can develop a discontinuity even if the initial-value function f is continuous. The basic idea is simple: If the wave velocity is u, then an initial value of u that is greater on the left side of a particular z than on the right side of the same z will create waves that travel faster on the left side of z than on the right side. The fast waves will overtake the slow waves, causing a discontinuity.

For example, let

$$f(z) = \begin{cases} 1, & \text{if } z \le 0; \\ 1 - z, & \text{if } 0 \le z \le 1; \\ 0, & \text{if } z \ge 1. \end{cases} \tag{16.44}$$

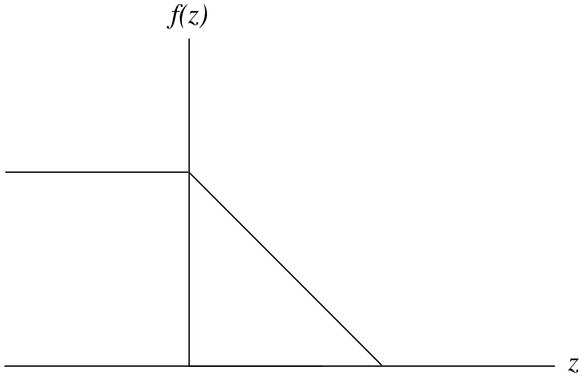

Figure 16.3: The initial velocity for Burgers' equation.

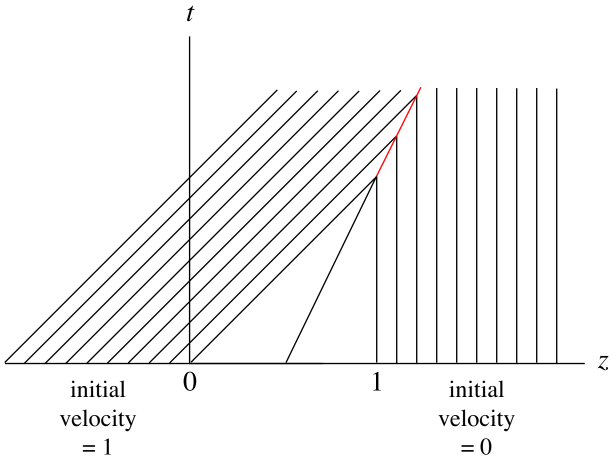

Figure 16.4: The characteristics for Burgers' equation. The red line indicates a shock front, along which u is discontinuous. To the left of the shock front, $u = 1$; to the right of the shock front, $u = 0$.

In this case, the initial velocity is 1 for negative z and 0 for $z \geq 1$ (see Fig. 16.3). The result that u is constant on characteristics that issue from the z-axis implies that the characteristics look as shown in Fig. 16.4.

From Fig. 16.4, it is clear that the waves that issue from the negative real axis overtake the waves that issue from the interval $(1, \infty)$ at a finite time. The function u is discontinuous at a point at which two characteristic curves intersect. To the left of the red line in Fig. 16.4, $u = 1$; to the right of the red line, $u = 0$. The red line itself is a **shock front**, which propagates at a finite velocity (not equal to the velocity of wave propagation), and along which u is discontinuous.

16.2.3 First-order, quasilinear, inhomogeneous PDEs

16.2.3.1 Method of characteristics

To obtain a solution of the inhomogeneous equation $a\partial u/\partial z + b\partial u/\partial t = c$, we return to Eqs. (16.15–16.21), which imply, together with the partial differential equation itself, that

$$\begin{aligned} \frac{du}{ds} &= \left(a\frac{\partial u}{\partial z} + b\frac{\partial u}{\partial t} \right) F(s) \\ &= c\,F(s). \end{aligned} \tag{16.45}$$

It follows from this equation and the identity $du/dt = (du/ds)/(dt/ds)$ that the general solution of the inhomogeneous first-order quasilinear partial differential equation (16.3) is

$$\boxed{\begin{aligned} \left(\frac{du}{dz} \right)_C &= \frac{c(u,z,t)}{a(u,z,t)} \\ \frac{dt}{dz} &= \frac{b(u,z,t)}{a(u,z,t)}. \end{aligned}} \tag{16.46}$$

If one solves this system of ordinary differential equations (numerically, for example), one simultaneously constructs the solution u and the **characteristic curve** C such that $dz/dt = a/b$. This **method of characteristics** reduces the partial differential equation () to a system of ordinary differential equations. The disadvantage of the method of characteristics is that it gives the solution u as a function on a characteristic curve, not as a function on the points of a rectangular grid in $\mathbb{R}^2$.

A traditional way to present the method of characteristics for first-order, quasilinear, inhomogeneous partial differential equations is due to Lagrange:

$$\boxed{\frac{dz}{a} = \frac{dt}{b} = \frac{du}{c}} \tag{16.47}$$

The ordinary differential equations that can be formed by making derivatives from quotients of the differentials dz, dt and du in Lagrange's method are equivalent to Eq. (16.46).

16.2.3.2 Numerical solution

In order to solve Eq. (16.26) or Eq. (16.46) numerically, one can use any method that is stable and accurate. For example, for the beam-propagation problem, the characteristic curves are straight lines $\big($Eq. (16.29)$\big)$, and therefore the only differential equation that needs to be solved is

$$\left(\frac{du}{dz'}\right)_C = \frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t} = i\gamma |u|^2 u, \tag{16.48}$$

where u is considered as a function of z' and t' $\big($Eq. (16.35)$\big)$. The initial data that must be given is the time dependence of u at $z = 0$, *i.e.*, $u(0, t')$. The list of appropriate finite-difference methods for this problem includes the midpoint-trapezoidal predictor-corrector, which my group has used for this and other laser-pulse propagation problems for many years.

The midpoint-trapezoidal predictor-corrector algorithm for Eq. (16.48) reads as follows, where $c_{n,m}$ is the computed value of $u(z'_n, t'_m)$:

$$\begin{aligned} &\text{predictor:} \quad p_{n+1,m} = c_{n-1,m} + 2ih\gamma |c_{n,m}|^2 c_{n,m} \\ &\text{corrector:} \quad c_{n+1,m} = c_{n,m} + i\frac{h}{2}\gamma\big(|p_{n+1,m}|^2 p_{n+1,m} + |c_{n,m}|^2 c_{n,m}\big). \end{aligned} \tag{16.49}$$

The computed values are complex.

Fig. 16.5 shows the Fourier transform of the computed function $u(z', t')$ with respect to t' for selected values of z'. In this example, the total number of steps in z' was 4×10^6; the step size in z' was 1 cm; and the number of values of t' was 1024. The computed function u shows substantial spectral broadening due to self-phase modulation.

Eqs. (16.26) and (16.46) gives u as a function defined on a characteristic curve and not as a function defined on $\mathbb{R}^2$. In order to compute the value $u(z, t)$ for given values of z and t for use in other applications such as the discrete Fourier transform, one must first compute u along a finite set of characteristic curves, and then approximate $u(z, t)$ by interpolating between points on different characteristic curves.

16.3 Second-order, quasilinear PDEs

16.3.1 Classification

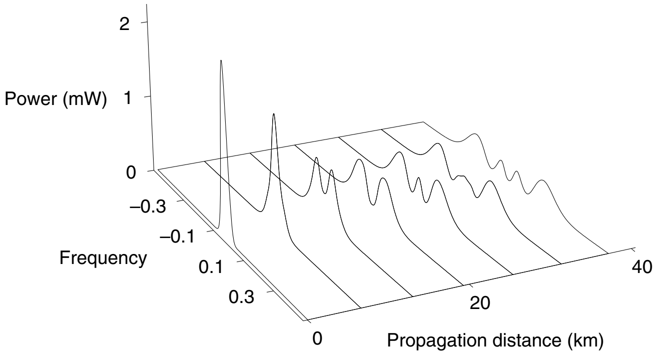

Figure 16.5: Fourier transform of $u(z', t')$ with respect to t' for selected values of z'. Courtesy of D. M. Hollenbeck.

16.3.1.1 Definitions and motivation

The most general second-order quasilinear PDE in two independent variables is

$$\begin{gathered} a(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x^2} + b(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x \partial y} \\ + c(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial y^2} + e(x,y,u,u_x,u_y) = 0 \end{gathered} \tag{16.50}$$

where $u_x = \partial u/\partial x$, etc. Solving such an equation means finding a numerical or analytic function u (and its derivatives) as functions of x and y in the plane, given values of u (and a derivative of u that we will specify below) on a curve Γ. In order to discover what data we need along Γ, let us assume that we know u (and all of its derivative) on Γ, let s be the arc length along Γ, and let

$$\mathbf{s} = \begin{pmatrix} \dfrac{dx}{ds} \\ \\ \dfrac{dy}{ds} \end{pmatrix} \tag{16.51}$$

be the tangent vector to Γ at s. As is true for $\mathbf{t}$, Eq. (16.19), $\mathbf{s}$ is a unit vector. Since we know u along Γ, we can compute the derivative of u along Γ,

$$\frac{du}{ds} = \mathbf{s} \cdot \nabla u = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}. \tag{16.52}$$

In this equation, we know the values of dx/ds and dy/ds, because we have assumed that we know how to define Γ. However, we do not know the values of the first partial derivatives of u. If we knew the values of $\partial u/\partial x$ and $\partial u/\partial y$, then we could find the value of u at a point $(x + \Delta x, y + \Delta y)$ that is close to a point (x, y) on Γ by using the approximation

$$u(x + \Delta x, y + \Delta y) \approx u(x, y) + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y. \tag{16.53}$$

Our first goal, then, is to compute $\partial u/\partial x$ and $\partial u/\partial y$ given data along Γ such as u. Of course, the value of the function u and the values of its first partial derivatives are not enough to compute u at points that are far from Γ. To do that, it is enough to know *all* of the partial derivatives of u, because that permits us to compute u via a power series expansion:

$$u(x',y') = u(x,y) + \frac{\partial u}{\partial x}(x'-x) + \frac{\partial u}{\partial y}(y'-y) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x'-x)^2 + \cdots \tag{16.54}$$

In this equation, (x,y) is a point on Γ, and (x',y') is a point in the plane, but not on Γ.

16.3.1.2 Determination of the first partial derivatives of u

It turns out that, if we also know the normal derivative, du/dn, of u along Γ, then we can compute the first partial derivatives of u. The unit vector that is normal to Γ at s is

$$\mathbf{n} = \begin{pmatrix} -\dfrac{dy}{ds} \\ \dfrac{dx}{ds} \end{pmatrix}. \tag{16.55}$$

(Reader: Please verify that $\mathbf{n}\cdot\mathbf{s} = 0$ before reading further!) Therefore the normal derivative of u at s is

$$\frac{du}{dn} = \mathbf{n}\cdot\nabla u = -\frac{\partial u}{\partial x}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds}. \tag{16.56}$$

The system of linear equations

$$\begin{aligned} \frac{du}{ds} &= \frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dy}{ds}\frac{\partial u}{\partial y} \\ \frac{du}{dn} &= -\frac{dy}{ds}\frac{\partial u}{\partial x} + \frac{dx}{ds}\frac{\partial u}{\partial y} \end{aligned} \tag{16.57}$$

can be solved for the unknowns $\partial u/\partial x$ and $\partial u/\partial y$ provided that the determinant of coefficients is non-zero. The determinant of the coefficients of $\partial u/\partial x$ and $\partial u/\partial y$ is

$$\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} \\ -\dfrac{dy}{ds} & \dfrac{dx}{ds} \end{vmatrix} = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1, \tag{16.58}$$

which is certainly non-zero. Therefore the unique solution of the linear system (16.57) is

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} \frac{du}{ds} & \frac{dy}{ds} \\ \frac{du}{dn} & \frac{dx}{ds} \end{vmatrix}}{\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ -\frac{dy}{ds} & \frac{dx}{ds} \end{vmatrix}} = \frac{du}{ds}\frac{dx}{ds} - \frac{du}{dn}\frac{dy}{ds} \tag{16.59}$$

and

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} \frac{dx}{ds} & \frac{du}{ds} \\ -\frac{dy}{ds} & \frac{du}{dn} \end{vmatrix}}{\begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ -\frac{dy}{ds} & \frac{dx}{ds} \end{vmatrix}} = \frac{du}{ds}\frac{dy}{ds} + \frac{du}{dn}\frac{dx}{ds}. \tag{16.60}$$

Now that we have succeeded in computing the first partial derivatives given the values of u and du/dn on Γ, we will try to compute partial derivatives of higher order.

16.3.1.3 Boundary and initial conditions

Boundary conditions such that the values of u and du/dn are given on a boundary curve Γ are called **Cauchy conditions**. Other, weaker types of boundary conditions are **Dirichlet conditions**, in which only the value of u is given on Γ, and **Neumann conditions**, in which only the value of du/dn is given on Γ.

16.3.1.4 Characteristic equation

There are three equations that we can use to determine the three second partial derivatives of u given u and du/dn on Γ, the first partial derivatives that we just computed, and the partial differential equation itself:

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) &= \frac{dx}{ds}\frac{\partial^2 u}{\partial x^2} + \frac{dy}{ds}\frac{\partial^2 u}{\partial x \partial y} \\ \frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) &= \frac{dx}{ds}\frac{\partial^2 u}{\partial x \partial y} + \frac{dy}{ds}\frac{\partial^2 u}{\partial y^2} \\ -e &= a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} \end{aligned} \tag{16.61}$$

This system of three linear equations in the three unknowns $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x\partial y$, and $\partial^2 u/\partial y^2$ has a unique solution if and only if the determinant of coefficients,

$$\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} & 0 \\ 0 & \dfrac{dx}{ds} & \dfrac{dy}{ds} \\ a & b & c \end{vmatrix} = a\left(\frac{dy}{ds}\right)^2 - b\frac{dy}{ds}\frac{dx}{ds} + c\left(\frac{dx}{ds}\right)^2, \tag{16.62}$$

is non-zero. To make the situation when this condition fails easier to analyze, we take advantage of the fact that, along Γ,

$$\frac{dy}{dx} = \frac{dy}{ds}\Big/\frac{dx}{ds}. \tag{16.63}$$

After we multiply the right-hand side of (16.62) by $(dx/ds)^{-2}$, then, we obtain

$$\boxed{a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c = 0} \tag{16.64}$$

as the condition for the vanishing of the determinant of the coefficients of the linear system (16.61). Eq. (16.64), the **characteristic equation** of the PDE (16.50), is the condition that there is *not* a unique solution for the second-order partial derivatives in terms of data on the curve Γ.

The roots of the characteristic equation (16.64) are

$$\frac{dy}{dx} = \lambda = \frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}. \tag{16.65}$$

Each distinct root defines a differential equation, the solution of which is a family of **characteristic curves** C in the x–y plane. We have shown that it is not possible to compute the partial derivatives of second order given only data on a characteristic curve.

16.3.1.5 Classification

There are three different types of second-order quasilinear PDEs in two independent variables, depending on the nature of the roots given in Eq. (16.65):

$$\text{If } b^2 - 4ac \text{ is } \begin{cases} > 0, & \text{the PDE is } \textbf{hyperbolic}; \\ = 0, & \text{the PDE is } \textbf{parabolic}; \\ < 0, & \text{the PDE is } \textbf{elliptic}. \end{cases} \tag{16.66}$$

$b^2 - 4ac > 0$: The characteristic equation has two real roots. There are two real, distinct families of characteristics. Example: The wave equation,

$$\frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{16.67}$$

in which $a = 1$, $b = 0$, $c = -1/v^2$, and $b^2 - 4ac = 4/v^2$. For this example, it is possible to change variables in such a way that only b is non-zero in the transformed PDE. Hyperbolic equations require Cauchy initial conditions.

$b^2 - 4ac = 0$: The characteristic equation has one real root, equal to zero. There is one real family of characteristics. Example: The diffusion equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{16.68}$$

in which $a = 1$, $b = 0$, $c = 0$, $b^2 - 4ac = 0$, and the root is zero. Parabolic equations require either Dirichlet or Neumann initial conditions.

$b^2 - 4ac > 0$: The characteristic equation has two complex conjugate roots. There are no real characteristics. Example: Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{16.69}$$

in which $a = 1$, $b = 0$, $c = 1$, and $b^2 - 4ac = -4$. Elliptic equations require either Dirichlet or Neumann boundary conditions.

16.3.2 Hyperbolic PDEs

16.3.2.1 Characteristic differential equations

The goal here is to express the original hyperbolic PDE as two differential equations for the characteristic curves, plus a differential equation that describes how u changes along a curve.

From the first two equations of (16.61) one can solve for $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ in terms of $\partial^2 u/\partial x \partial y$, obtaining

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\dfrac{d}{ds}\left(\dfrac{\partial u}{\partial x}\right) - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dy}{ds}}{\dfrac{dx}{ds}} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\dfrac{d}{ds}\left(\dfrac{\partial u}{\partial y}\right) - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dx}{ds}}{\dfrac{dy}{ds}}.\end{aligned} \tag{16.70}$$

In order to eliminate $\partial^2 u/\partial x \partial y$, we substitute these expressions into the PDE:

$$a\frac{\dfrac{d}{ds}\left(\dfrac{\partial u}{\partial x}\right) - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dy}{ds}}{\dfrac{dx}{ds}} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\dfrac{d}{ds}\left(\dfrac{\partial u}{\partial y}\right) - \dfrac{\partial^2 u}{\partial x \partial y}\dfrac{dx}{ds}}{\dfrac{dy}{ds}} = -e. \tag{16.71}$$

Clearing by multiplying through by $(dx/ds)(dy/ds)$ and then collecting terms, one gets

$$\begin{aligned}&\frac{\partial^2 u}{\partial x \partial y}\left[-a\left(\frac{dy}{ds}\right)^2+b\frac{dx}{ds}\frac{dy}{ds}-c\left(\frac{dx}{ds}\right)^2\right]\\&\qquad=\left[a\frac{dy}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right)+c\frac{dx}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right)+e\frac{dx}{ds}\frac{dy}{ds}\right].\end{aligned} \tag{16.72}$$

The first line of Eq. (16.72) vanishes on a characteristic curve C. Therefore, on C, both of the equations

$$\begin{aligned}a\frac{dy}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right)+c\frac{dx}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right)+e\frac{dx}{ds}\frac{dy}{ds}=0\\ \text{and}\quad \frac{dy}{ds}=\lambda\frac{dx}{ds},\end{aligned} \tag{16.73}$$

must be obeyed, where λ is one of the roots of the characteristic equation (16.64) given in Eq. (16.65). Substituting for dy/ds and cancelling dx/ds, one obtains the **characteristic differential equations** (for $i = 1, 2$)

$$\boxed{\lambda_i\frac{a}{c}\left(\frac{d}{ds}\right)_i\left(\frac{\partial u}{\partial x}\right)+\left(\frac{d}{ds}\right)_i\left(\frac{\partial u}{\partial y}\right)=-\lambda_i\frac{e}{c}\left(\frac{dx}{ds}\right)_i.} \tag{16.74}$$

The ordinary differential equation

$$\boxed{\frac{du}{ds}=\frac{dx}{ds}\frac{\partial u}{\partial x}+\frac{dy}{ds}\frac{\partial u}{\partial y}} \tag{16.75}$$

which describes the variation of u along any curve, together with Eqs. (16.74), determine u, given Cauchy initial conditions.

16.3.2.2 The one-dimensional wave equation

The **one-dimensional wave equation** is

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)u(z,\,t)=0. \tag{16.76}$$

For this equation, $a = 1$, $b = 0$, $c = -1/v^2$, and $e = 0$, making the eigenvalue equation (16.65) read

$$\frac{dt}{dz}=\lambda=\pm\frac{1}{2}\sqrt{4/v^2}=\pm\frac{1}{v}. \tag{16.77}$$

Therefore the equations of the characteristic curves are

$$z \pm vt = k \tag{16.78}$$

where k is any real number.

Let

$$\xi = z + vt, \quad \eta = z - vt. \tag{16.79}$$

The equations $\xi = k$ and $\eta = k$ are the equations of characteristic curves. As is true for hyperbolic PDEs in general, there are two families of characteristic curves:

- Family 1, consisting of the lines $\eta = z - vt = \text{constant}$
- Family 2, consisting of the lines $\xi = z + vt = \text{constant}$

The element of arc length measured along a characteristic curve $z \pm vt = k$ obeys the equation $ds^2 = dz^2 + (vdt)^2$. Therefore

$$ds = \pm\, dz \left[1 + v^2 \left(\frac{dt}{dz}\right)^2\right]^{\frac{1}{2}} = \pm\sqrt{2}\, dz. \tag{16.80}$$

For family 1,

$$(ds)_1 = \sqrt{2}\, dz = \frac{1}{\sqrt{2}}\, d\xi, \quad \lambda = \lambda_1 = \frac{1}{v}; \tag{16.81}$$

for family 2,

$$(ds)_2 = -\sqrt{2}\, dz = -\frac{1}{\sqrt{2}}\, d\eta, \quad \lambda = \lambda_2 = -\frac{1}{v}. \tag{16.82}$$

The **canonical variables** ξ, η, or, equivalently, the **characteristic variables**

$$\begin{aligned} s_1 &= \frac{1}{\sqrt{2}}\,\xi\,, \\ s_2 &= -\frac{1}{\sqrt{2}}\,\eta\,, \end{aligned} \tag{16.83}$$

define a coordinate net on which the wave equation takes on an especially simple form.

16.3.2.3 Solution of the wave equation by the method of characteristics

To express Eq. (16.76) in terms of canonical or characteristic variables, we make use of the chain rule for partial differentiation, which implies that

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{1}{v}\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}. \tag{16.84}$$

Then

$$\frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right), \quad \frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}\right). \tag{16.85}$$

Recalling that $a = 1$, $b = 0$, $c = -1/v^2$, and $e = 0$, we see that for the one-dimensional wave equation the characteristic differential equations (16.74) take on the forms

$$\left(\frac{d}{ds}\right)_1 \left(\frac{\partial u}{\partial z} - \frac{1}{v}\frac{\partial u}{\partial t}\right) = 0, \quad \left(\frac{d}{ds}\right)_2 \left(\frac{\partial u}{\partial z} + \frac{1}{v}\frac{\partial u}{\partial t}\right) = 0. \tag{16.86}$$

Let

$$w(\xi,\, \eta) := u(z,\, t). \tag{16.87}$$

With Eqs. (16.85) and (16.87), the characteristic differential equations become

$$\frac{d}{d\xi}\left(\frac{\partial w}{\partial \eta}\right) = 0, \quad \frac{d}{d\eta}\left(\frac{\partial w}{\partial \xi}\right) = 0. \tag{16.88}$$

One can see at once that the general solution of Eqs. (16.88) is

$$\frac{\partial w}{\partial \eta} = f'(\eta), \quad \frac{\partial w}{\partial \xi} = g'(\xi) \tag{16.89}$$

where f and g are arbitrary continuously twice-differentiable functions. Therefore the general solution of Eq. (16.76) is

$$w(\xi, \eta) = f(\eta) + g(\xi) \Rightarrow u(z,\, t) = f(z - vt) + g(z + vt). \tag{16.90}$$

The wave represented by f travels in the $+z$ direction, since f takes on the same value, $f(\xi_0)$, for every spacetime point $(z,\, t)$ such that $z - vt = \xi_0$. Similarly, the wave represented by g travels in the $-z$ direction.

16.3.2.4 Solution for a string of infinite length

For a string of infinite length two initial conditions are necessary in order to determine the two independent functions f and g in Eq. (16.90). Let

$$d(z) := u(z,\, 0), \quad s(z) := \frac{1}{v}\frac{\partial u}{\partial t}(z,\, t)\bigg|_{t=0}. \tag{16.91}$$

The function d specifies the initial displacement profile of the wave, while the function s specifies the x-component of the velocity of the string itself, expressed as a fraction of the phase velocity of waves on the string. It follows that

$$d(z) = f(z) + g(z) \tag{16.92}$$

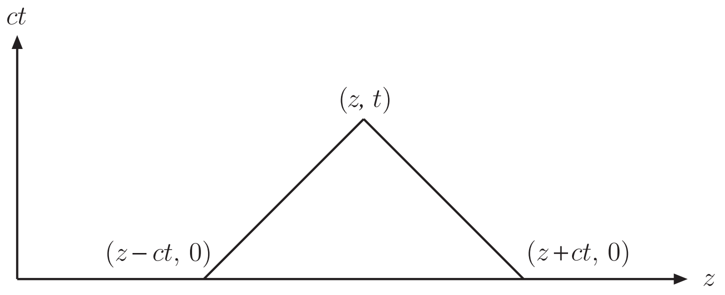

Figure 16.6: The domain of dependence for the wave equation on an infinitely long string.

and that

$$s(z) = -f'(z) + g'(z) \Rightarrow -f(z) + g(z) = v \int_0^z s(z')\, dz'. \tag{16.93}$$

From Eq. (16.90) and Eqs. (16.91–16.93) one finds that the general solution of the wave equation for an infinitely long string is

$$u(z,\, t) = \tfrac{1}{2}\left[d(z - vt) + d(z + vt) + v \int_{z-vt}^{z+vt} s(z')\, dz'\right]. \tag{16.94}$$

Fig. 16.6 illustrates the fact that the value of u at the spacetime point $(z,\, t)$ depends upon the values of d at the points $(z \pm vt,\, 0)$ and the values of $\partial u/\partial t$ on the interval $[z - vt,\, z + vt]$ along the z-axis. The straight lines with slopes ± 1 which intersect at $(z,\, t)$ are examples of *characteristic curves* for the wave equation Eq. (16.76). The triangular spacetime region defined by the points $(z,\, t)$ and $(z \pm vt,\, 0)$ is called the **domain of dependence**, because it can be shown that the value of u at $(z,\, t)$ depends upon $\partial u/\partial t$ along any curve which traverses this domain and intersects both characteristic curves.

Because of its very general form, the d'Alembert solution imposes no restrictions on the frequencies which may be present in a one-dimensional wave.

16.3.2.5 Solution for a string of finite length

We shall show that the boundary conditions for a finite string given in Eq. (**??**) imply that there is only one unknown function to be determined in the d'Alembert solution Eq. (16.90), and that that function is periodic with a period equal to twice the length of the string. The boundary condition $u(0,\, t) = 0$ implies that

$$\forall\, t \in \mathbb{R} : f(0 - vt) + g(0 + vt) = 0. \tag{16.95}$$

Therefore g is simply f, reflected in the origin and reversed in sign:

$$\forall\, z \in \mathbb{R} : g(z) = -f(-z). \tag{16.96}$$

From a physical point of view it is obvious that the backward-traveling wave $g(z + vt)$ can exactly cancel the forward-traveling wave $f(z - vt)$ at $z = 0$ for all times t only if Eq. (16.96) is obeyed.

Eq. (16.96) and the boundary condition $u(L, t) = 0$ imply that

$$\forall\, t \in \mathbb{R} : f(L - vt) - f(-L - vt) = 0. \tag{16.97}$$

After substituting $vt = -L - z$ one sees that f is periodic with period $2L$:

$$\forall\, z \in \mathbb{R} : f(z + 2L) = f(z). \tag{16.98}$$

Therefore f possesses a Fourier series on an interval of length $2L$:

$$f(z) = \sum_{m=-\infty}^{\infty} f_m e^{2\pi i m z/2L}. \tag{16.99}$$

From Eq. (16.96) it follows that

$$g(z) = -\sum_{m=-\infty}^{\infty} f_m e^{-im\pi z/L}. \tag{16.100}$$

Then Eq. (16.90) implies that

$$\begin{aligned} u(z,\, t) &= \sum_{m=-\infty}^{\infty} f_m \left(e^{im\pi(z-vt)/L} - e^{-im\pi(z+vt)/L} \right) \\ &= 2i \sum_{m=-\infty}^{\infty} f_m\, e^{-im\pi vt/L} \sin(m\pi z/L). \end{aligned} \tag{16.101}$$

The sine function guarantees that u vanishes at $z = 0$ and $z = L$.

The condition that u must be real-valued, $u(z,\, t)^* = u(z,\, t)$, implies that

$$f_m^* = f_{-m}. \tag{16.102}$$

Then

$$u(z,\, t) = 2i \sum_{m=1}^{\infty} \left(f_m e^{-im\pi vt/L} - f_{-m} e^{im\pi vt/L} \right) \sin(m\pi z/L). \tag{16.103}$$

The sum runs from 1 to ∞ since $\sin(m\pi z/L) = 0$ when $m = 0$. Letting

$$f_m = |f_m| e^{-i\phi_m}, \tag{16.104}$$

one finds that the general solution to the wave equation for a string of length L is

$$u(z,\, t) = 4 \sum_{m=1}^{\infty} |f_m| \sin(m\pi z/L) \sin(m\pi vt/L + \phi_m). \tag{16.105}$$

The mutually orthogonal functions $\sin(m\pi z/L)$ are called the **normal modes** of the vibrating string. Each mode is a standing wave, that is, a wave such that the zeros (or nodes) occur at time-independent positions.

It is noteworthy that because the Dirichlet boundary conditions Eq. (**??**) imply that f must be periodic, the only frequencies which can appear in the solution are the frequencies of the normal modes. In this example the normal-mode frequencies are integer multiples of the **fundamental frequency** $v/2L$. Different boundary conditions give different normal-mode frequencies. For example, the mixed boundary conditions

$$u(0,\, t) = 0, \quad \frac{\partial u}{\partial z}(L,\, t) = 0 \tag{16.106}$$

give normal-mode frequencies which are odd multiples of $v/4L$.

The initial displacement of the string is

$$u(z,\, 0) = 4 \sum_{m=1}^{\infty} |f_m| \sin\phi_m \sin(m\pi z/L). \tag{16.107}$$

The coefficients of this Fourier sine series, $b_m = 4|f_m|\sin\phi_m$, can be determined uniquely by making use of the orthogonality of the sines of different multiples of the same angle. Therefore specifying the initial displacement determines a unique solution of the wave equation for all future times on a string of finite length.