Auction form preference and inefficiency of asymmetric discriminatory auctions

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Abstract

We examine the efficiency and expected revenue properties of an asymmetric discriminatory auction with two bidders and two objects. While inherently inefficient, the discriminatory auction may result in higher expected revenue than the efficient Vickrey mechanism.

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1 Introduction

In this paper we investigate the revenue properties of multi-unit discriminatory auctions with asymmetric bidders. We specifically consider a two-bidder, two-object environment characterized by one-sided incomplete information. A distinguishing feature of our model is the asymmetry between the two bidders' demands. Our structure guarantees that a particular Bayesian Nash equilibrium of a simple discriminatory auction exists for all continuous and strictly increasing distributions of valuations. We characterize this equilibrium.

The few papers in the literature on asymmetric auctions focus on the breakdown of revenue equivalence in the case of single-unit auctions. Maskin and Riley [3], for instance, show that expected revenue from the high-bid and open auctions is not the same when bidder symmetry is dropped. Landsberger *et. al.* [2] elaborate on an example in which bidder asymmetry is induced by a known ranking of their valuations. With uniformly distributed valuations, they show that bidders in this setting bid more aggressively than in a standard (symmetric) first-price auction. Recent work by Estelle Cantillon [1] shows that competition is hampered by bidder asymmetry in the first-price auction.

The assignment of the goods in our equilibrium is ex-ante inefficient. The source of inefficiency consists of both the asymmetry of information between bidders and the asymmetry of demand; specifically, a bidder with demand for two objects responds to a "monopsony" effect, which gives her an incentive to submit low bids for all objects to ensure low payment for objects she is likely to win irrespective of the bids of other bidders. We elaborate on this incentive in the body of the paper. We show using a parameterized class of environments that the inefficient discriminatory auction may raise more expected revenue than an efficient auction; this occurs exactly when the expected inefficiency of the discriminatory auction is relatively small.

2 The model

Two bidders participate in an auction of two identical objects. The first ("high-demand") bidder has constant marginal valuation θ for each object, distributed according to a cumulative distribution function $F(\theta)$, continuous and strictly increasing on $[0, 1]$. The second ("low-demand") bidder wishes to purchase only one object, for which he has valuation $\alpha \in (0, 1]$. $F(\theta)$ and α are common knowledge and only the high-demand bidder observes the realization of θ. The high-demand bidder submits two bids and the low-demand bidder one bid. If the high-demand bidder submits the highest bids she wins both objects and pays the sum of her bids; if the low-demand bidder submits the highest or second-highest bid, he wins one object and pays his bid, while the high-demand bidder wins the remaining object and pays the greater of her two bids.

In any Bayesian Nash equilibrium of this game, any type of high-demand bidder must submit two equal bids. To see this, note that for any strategy profile, the high-demand bidder will win at least one object and will always pay her greater bid for that object. Furthermore, reducing her greater bid has no effect on the probability that she wins both objects. Thus, for any two pairs of bids (b', b) and (b, b) with $b' > b$, and for any strategy for the low-demand bidder, the high-demand bidder earns a strictly greater payoff by bidding (b, b). We summarize the pair (b, b) by b and refer to it as the high-demand bidder's "bid."

We are able to characterize a Bayesian Nash equilibrium of this auction. In this equilibrium, the high-demand bidder's strategy $\beta(\theta)$ calls for her to bid 0 if her type θ falls below an endogenously determined level $\tilde{\theta}$, and otherwise to make bids that are strictly increasing in θ. The low-demand bidder uses a mixed strategy, characterized by a continuous distribution $H(b)$ over a connected support bounded below by 0.

Proposition 1 *For any continuous and strictly increasing distribution* $F(\cdot)$ *with support on* $[0,1]$ *and for any* $\alpha \in (0,1]$*, an equilibrium in the discriminatory auction is characterized by:*

$$\beta(\theta) = \begin{cases} 0 & \text{if } \theta \leq \tilde{\theta} \\ \alpha\left(1 - \frac{F(\tilde{\theta})}{F(\theta)}\right) & \text{if } 1 \geq \theta > \tilde{\theta} \end{cases} \tag{1}$$

and

$$H(b) = \begin{cases} \exp\left[\int_0^b \frac{dx}{F^{-1}\left(\frac{\alpha F(\tilde{\theta})}{\alpha - x}\right) - x}\right] - 1, & \text{if } b \in [0, \alpha(1 - F(\tilde{\theta}))] \\ 1 \text{ if } b \geq \alpha(1 - F(\tilde{\theta})) \end{cases}, \tag{2}$$

where $\tilde{\theta} \in (0,1)$ *and satisfies* $H(\alpha(1 - F(\tilde{\theta}))) = 1$.

Proof. See Appendix. ■

2.1 Discussion

As in equilibria of single-object first-price auctions with asymmetric bidders and private information, the equilibrium identified above leads to an inefficient allocation of the objects with positive probability. The source of inefficiency in a single-object auction is private information; in our setting, asymmetric demand also contributes to inefficiency. When valuations are common knowledge, there is no inefficiency in a single-object auction: the bidder with the highest valuation wins the object by bidding the second-highest valuation in any equilibrium. However, in our setting, suppose the high-demand bidder values each object at α', where $\alpha' > \alpha > \alpha'/2$ and α' is common knowledge. It is efficient for the high-demand bidder to win both objects, but in equilibrium the low-demand bidder must win an object with positive probability: if not, the high-demand bidder would have to bid at least α with probability 1, but since $2(\alpha' - \alpha) < \alpha'$, the high-demand bidder would prefer to bid 0 and win one object than bid α and win two objects. That is, the high-demand bidder's incentive to reduce his bids creates an extra potential source of inefficiency. We dub this incentive a "monopsony effect," as it is similar to a monopsonist who demands a low quantity to buy at a lower price.

Because the two-object discriminatory auction equilibrium is inefficient, the expected revenue it generates will differ in general from the revenue generated by an efficient auction. We compare here the revenue from the discriminatory auction to the revenue from a Vickrey auction, which is known to generate the highest revenue in the class of efficient mechanisms. A Vickrey auction in our two-object environment allocates one object to the informed bidder for free, while the second object is allocated using the rules of a second-price auction. As always, the Vickrey auction allocates the objects efficiently.

To compare revenue from the discriminatory and Vickrey two-object auctions, we consider a particular one-dimensional class of environments that enables analytic solutions for equilibria and revenue. Specifically, let the low-demand bidder's valuation α lie in $(1/3, 1]$, and the distribution of the high-demand bidder's valuations be given by

$$F(\theta) = \frac{6\alpha - 2}{6\alpha + 1 - 3\theta}$$

for $1/3 \leq \theta \leq 1$, with $F(\theta)$ left unspecified for $\theta < 1/3$, but chosen so that $F(\theta)$ satisfies our continuity and strict monotonicity assumptions for all θ. Irrespective of the specification of $F(\theta)$ for

$\theta < 1/3$, the equilibrium in the discriminatory auction has the low-demand bidder mixing uniformly over bids in the interval $[0, 1/3]$ and the high-demand bidder bidding according to $\beta(\theta) = (3\theta - 1)/6$ for $\theta \in [1/3, 1]$ and 0 otherwise. Expected revenue from the discriminatory auction, as a function of α, equals

$$\frac{1+3\alpha}{3}\left[1-(3\alpha-1)\ln\left(\frac{3\alpha}{3\alpha-1}\right)\right].$$

Unlike the revenue from the discriminatory auction, revenue from the Vickrey auction does depend on the specification of $F(\theta)$ for $\theta < 1/3$. For comparison it is useful to bound the Vickrey revenue over all $F(\cdot)$ in the class considered. The expectation of θ conditional on $\theta \leq 1/3$, denoted below by X, lies between 0 and 1/3 for all $F(\cdot)$ in our class of distributions. Expected revenue from the Vickrey auction is

$$\frac{3\alpha(1-\alpha)}{3\alpha+1}+\frac{1}{9}\left[\frac{6\alpha+1}{3\alpha+1}-\frac{6\alpha+1}{6\alpha}-\ln\left(\frac{6\alpha}{3\alpha+1}\right)\right]+(1-\frac{1}{3\alpha})X.$$

Thus, expected revenue from the Vickrey auction can be bounded above by replacing X with 1/3, and bounded below by replacing X with 0. Figure 1 displays expected revenue from the discriminatory auction, as well as the upper and lower bounds on Vickrey revenue, as functions of α.

[INSERT FIGURE 1 ABOUT HERE]

We offer some insight into these results. In the discriminatory auction, since the strategies of the two bidders are invariant to α, the only effect of α on revenue is through the distribution of the informed bidder's type θ, which decreases stochastically in α. Thus, expected revenue from the discriminatory auction in our example is strictly decreasing in α. Furthermore, as α approaches 1/3 from above, $F(\cdot)$ approaches a degenerate distribution that puts probability 1 on $\theta = 1$. Since $\beta(1) = 1/3$, and since type $\theta = 1$ wins two objects with probability 1 in this equilibrium, expected revenue from the discriminatory auction in the limit as α approaches 1/3 is 2/3. In contrast, in the Vickrey auction, as α approaches 1/3, the probability that the informed bidder wins the second object at a price of α approaches 1, so Vickrey revenue approaches 1/3. Thus, there is an open region of values of α near 1/3 in which the discriminatory auction raises more revenue than the Vickrey auction.

As α increases above 1/3, Vickrey revenue increases locally for all specifications of $F(\cdot)$ that are strictly monotonic (so that $X > 0$). Intuitively, since revenue in the Vickrey auction is the smaller realization of two independent random variables (α here is degenerate), it benefits from distributions of the two variables that are not too extreme. As α goes to 1/3 the uninformed bidder's valuation becomes small, while as α goes to 1 the informed bidder's valuation becomes small; Vickrey revenue peaks for some intermediate value of α, and is a concave function of α. For high enough values of X, there are some intermediate values of α for which the Vickrey auction delivers greater expected revenue than the discriminatory auction.

The discriminatory auction outperforms the Vickrey auction in terms of expected revenue when the former is least inefficient. As α approaches 1/3 from above, the informed bidder wins both objects–an efficient outcome–with probability near 1. Similarly, as α increases (unboundedly, in principle), the probability that the uninformed bidder wins an object–again, an efficient outcome–approaches 1. Revenue from the discriminatory auction goes to 1/6 as α increases unboundedly, while revenue from the Vickrey auction goes to 0. This suggests a trade-off between the two auction forms: the Vickrey auction is efficient, and as such generates more surplus for the seller to capture. However, the discriminatory auction may engender more competition between the bidders for a given level of efficiency. The seller's preference for a particular auction form therefore appears to hinge on the degree to which the outcome of the discriminatory auction is inefficient.

3 Conclusion

Little is known about the effect of bidder asymmetry on the relationship between efficiency and expected revenue in multi-unit auctions, primarily because their analytic equilibria are hard to obtain. Our focus in this paper is on an analytically tractable auction environment where asymmetry concerns both the bidders' information and their quantities demanded. Both types of asymmetries contribute to the inefficient outcome of the discriminatory auction. However, comparison with a first price auction with similar information structure intimates that asymmetric bidder demands contribute decisively to the inefficiency of the discriminatory auction. Our model implies that discriminatory auctions may be plagued by a particular form of demand reduction (or "monopsony effect") whereby one of the bidders, by bidding zero for both objects, may concede an object to a quantity-constrained opponent who has a lower valuation.

While inherently inefficient, we show that the discriminatory auction may produce a higher expected revenue than the efficient Vickrey auction; therefore, if the seller is not concerned with efficiency, the discriminatory auction may be the preferred auction form. The good news, however, is that within the context of our example the expected revenue performance of the discriminatory auction relative to the Vickrey auction is best when the former is least inefficient.

References

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Appendix

Lemma A *For $\alpha \in (0,1)$ and $F(\cdot)$ strictly increasing and continuous on $[0,1]$, there exists a unique $\tilde{\theta}$ such that the following are true*

(i) $\tilde{\theta} \in (0,1)$

(ii) $F^{-1}\left(\alpha F(\tilde{\theta})(\alpha - x)^{-1}\right) - x > 0$ *for all* $x \in \left[0, \alpha(1 - F(\tilde{\theta}))\right]$

(iii) $\exp\left[\int_0^{\alpha(1-F(\tilde{\theta}))} \left(F^{-1}\left(\alpha F(\tilde{\theta})(\alpha - x)^{-1}\right) - x\right)^{-1} dx\right] - 1 = 1.$

Proof. Consider the function $g(x) = \alpha - (\alpha - x)F(x)$, defined for $x \in [0, \alpha]$. Observe that $g(0) = g(\alpha) = \alpha$, that $x < g(x) < \alpha$ whenever $0 < x < \alpha$ and that $g(\cdot)$ is continuous. Denote by $\mathcal{M} = \arg\min_{x \in [0,\alpha]} g(x)$. The set $\mathcal{M}$ is not empty and compact; let $x^* = \min(x : x \in \mathcal{M})$ and $\hat{b} = g(x^*)$ and note that $\alpha > \hat{b} > x^* > 0$. For each $t \in [0, \hat{b})$, define $G(t) = \int_0^t \frac{dx}{F^{-1}(\frac{\alpha-t}{\alpha-x})-x}$. The function $G(\cdot)$ is continuous and strictly increasing on $[0, \hat{b})$ and we next show that $G(\cdot)$ is unbounded as well.

Claim: For every $M > 0$, there exists a $\hat{t} \in [0, \hat{b})$ such that $G(\hat{t}) \geq M$.
Step 1: There exists $\xi \in (0, x^*)$ such that $\int_0^{x^*-\xi} \frac{dx}{F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x})-x} > M + 1$. To see why, note that the strict monotonicity of $F(\cdot)$ implies $F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x}) - F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x^*}) < 0$ for all $x \in (0, x^*)$. In addition, since $x \in (0, x^*)$, $F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x}) - x = \left[F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x}) - x\right] - \left[F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x^*}) - x^*\right] < x^* - x$. Therefore, for $x \in (0, x^*)$, $\frac{1}{F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x})-x} > \frac{1}{x^*-x}$. Choose ξ so that $0 < x^* - \sqrt{\xi} < x^* - \xi$ and $0 < \xi < \exp[-(2M+2)]$. Clearly,

$$\int_0^{x^*-\xi} \frac{dx}{F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x}) - x} \geq \int_{x^*-\sqrt{\xi}}^{x^*-\xi} \frac{dx}{F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x}) - x} \geq \int_{x^*-\sqrt{\xi}}^{x^*-\xi} \frac{dx}{x^* - x} = -(\ln\xi - \ln\sqrt{\xi}) = \frac{-\ln\xi}{2} > M+1.$$

Step 2: There exists $\hat{t} \in [0, \hat{b})$ such that $\frac{1}{F^{-1}(\frac{\alpha-\hat{t}}{\alpha-x})-x} \geq \left[\frac{1}{F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x})-x}\right] - 1$ whenever $0 \leq x \leq x^* - \xi$. Choose $\eta > 0$ so that $0 < x^* - \xi < x^* < \hat{b} - \eta < \hat{b}$ and define two compact sets C and D as follows: $C = [0, x^* - \xi]$ and $D = [\hat{b} - \eta, \hat{b}]$. The map $(x,t) \mapsto F^{-1}(\frac{\alpha-t}{\alpha-x}) - x$ is continuous and positive for each $(x,t) \in C \times D$. Therefore, the function $G : C \times D \to \mathbb{R}$ defined by $G(x,t) = \frac{1}{F^{-1}(\frac{\alpha-t}{\alpha-x})-x}$ is uniformly continuous on $C \times D$ since $C \times D$ is compact. Hence, there exists $\delta > 0$ such that $|G(x,t) - G(x',t')| < 1$ for $(x,t), (x',t') \in C \times D$ with $|x - x'| + |t - t'| < \delta$. In particular, $G(x,t) - G(x,\hat{b}) > -1$ whenever $x \in C$, $t \in D$ and $|t - \hat{b}| < \delta$ and the proof of the claim is complete. $\square$

For ξ and $\hat{t}$ defined above,

$$\begin{aligned}
\int_0^{\hat{t}} \frac{dx}{F^{-1}(\frac{\alpha-\hat{t}}{\alpha-x}) - x} &= \int_0^{x^*-\xi} \frac{dx}{F^{-1}(\frac{\alpha-\hat{t}}{\alpha-x}) - x} + \int_{x^*-\xi}^{\hat{t}} \frac{dx}{F^{-1}(\frac{\alpha-\hat{t}}{\alpha-x}) - x} \\
&\geq \int_0^{x^*-\xi} \frac{dx}{F^{-1}(\frac{\alpha-\hat{t}}{\alpha-x}) - x} \geq \int_0^{x^*-\xi} \left[\frac{dx}{F^{-1}(\frac{\alpha-\hat{b}}{\alpha-x}) - x}\right] - (x^* - \xi) > M
\end{aligned}$$

Since $G(\cdot)$ is continuous and strictly increasing on $[0, \hat{b})$, we can apply the claim and deduce the existence of $\overline{b} \in [0, \hat{b})$ such that $G(\overline{b}) = \ln 2$. Defining $\tilde{\theta} = F^{-1}(1 - \frac{\overline{b}}{a})$, it is easily established that (i), (ii) and (iii) are satisfied and the proof of the lemma is complete. ■

Proof of Proposition 1

Let $\tilde{\theta} \in (0, 1)$ satisfy the conditions of the Lemma.

Step 1: $H(\cdot)$ is a best response to $\beta(\cdot)$. To show this, let the uninformed bidder's payoff from a bid b be denoted $\varphi(b)$. Hence, $\varphi(b) = (\alpha - b)\,\mathrm{Prob}\{\beta(X) \leq b\}$ where $X \sim F$ and note that

$$\begin{aligned}\varphi(b) &= (\alpha - b)\left[\mathrm{Prob}\{\beta(X) \leq b, 0 \leq X \leq \tilde{\theta}\} + \mathrm{Prob}\{\beta(X) \leq b, \tilde{\theta} \leq X \leq 1\}\right] \\ &= (\alpha - b)\left[F(\tilde{\theta}) + \mathrm{Prob}\{\beta(X) \leq b, \tilde{\theta} \leq X \leq 1\}\right].\end{aligned}$$

If $b \in [0, \alpha(1 - F(\tilde{\theta}))]$, then

$$\mathrm{Prob}\{\beta(X) \leq b, \tilde{\theta} \leq X \leq 1\} = \mathrm{Prob}\{\tilde{\theta} \leq X \leq F^{-1}(F(\tilde{\theta})(\frac{\alpha}{\alpha - b}))\} = F(\tilde{\theta})(\frac{\alpha}{\alpha - b}) - F(\tilde{\theta})$$

and it follows that $\varphi(b) = \alpha F(\tilde{\theta})$. If $b \geq \alpha(1 - F(\tilde{\theta}))$, then $\mathrm{Prob}\{\beta(X) \leq b, \tilde{\theta} \leq X \leq 1\} = \mathrm{Prob}\{\tilde{\theta} \leq X \leq 1\} = 1 - F(\tilde{\theta})$ and it follows that $\varphi(b) = \alpha - b \leq \alpha F(\tilde{\theta})$.

Step 2: $\beta(\cdot)$ is a best response to $H(\cdot)$. Let $\psi(b|\theta)$ denote the payoff to an informed bidder of type θ from a bid b. Hence,

$$\psi(b|\theta) = 2(\theta - b)H(b) + (\theta - b)(1 - H(b)) = (\theta - b)(1 + H(b)).$$

First, observe that for each $\theta \in [0, 1]$,

$$\max_{b \geq \alpha(1 - F(\tilde{\theta}))} \psi(b|\theta) = \max_{b \geq \alpha(1 - F(\tilde{\theta}))} 2(\theta - b) = \psi(\alpha(1 - F(\tilde{\theta}))|\theta).$$

Next define, define $G(b) = \exp\left[\int_0^b \frac{dx}{F^{-1}(\frac{\alpha F(\tilde{\theta})}{\alpha - x}) - x}\right]$ for each $b \in [0, \alpha(1 - F(\tilde{\theta}))]$. If $b \in [0, \alpha(1 - F(\tilde{\theta}))]$,then (ii) of the Lemma implies that $\psi(b|\theta) = (\theta - b)G(b)$ and $\psi(\cdot|\theta)$ is differentiable with

$$\psi'(b|\theta) = \left[\frac{\theta - b}{F^{-1}(\frac{\alpha F(\tilde{\theta})}{\alpha - b}) - b} - 1\right] G(b).$$

Suppose that $\tilde{\theta} \leq \theta \leq 1$. It is easy to verify that $0 \leq \alpha(1 - \frac{F(\tilde{\theta})}{F(\theta)}) \leq \alpha(1 - F(\tilde{\theta}))$ and that $\psi'(\alpha(1 - \frac{F(\tilde{\theta})}{F(\theta)})|\theta) = 0$, $\psi'(b|\theta) \geq 0$ if $0 \leq b \leq \alpha(1 - \frac{F(\tilde{\theta})}{F(\theta)})$ and $\psi'(b|\theta) \leq 0$ if $\alpha(1 - \frac{F(\tilde{\theta})}{F(\theta)}) \leq b \leq \alpha(1 - F(\tilde{\theta}))$. Therefore,

$$\psi(\alpha(1 - \frac{F(\tilde{\theta})}{F(\theta)})|\theta) = \max_{b \in [0, \alpha(1 - F(\tilde{\theta}))]} \psi(b|\theta) \geq \psi(\alpha(1 - F(\tilde{\theta}))|\theta) = \max_{b \geq \alpha(1 - F(\tilde{\theta}))} \psi(b|\theta)$$

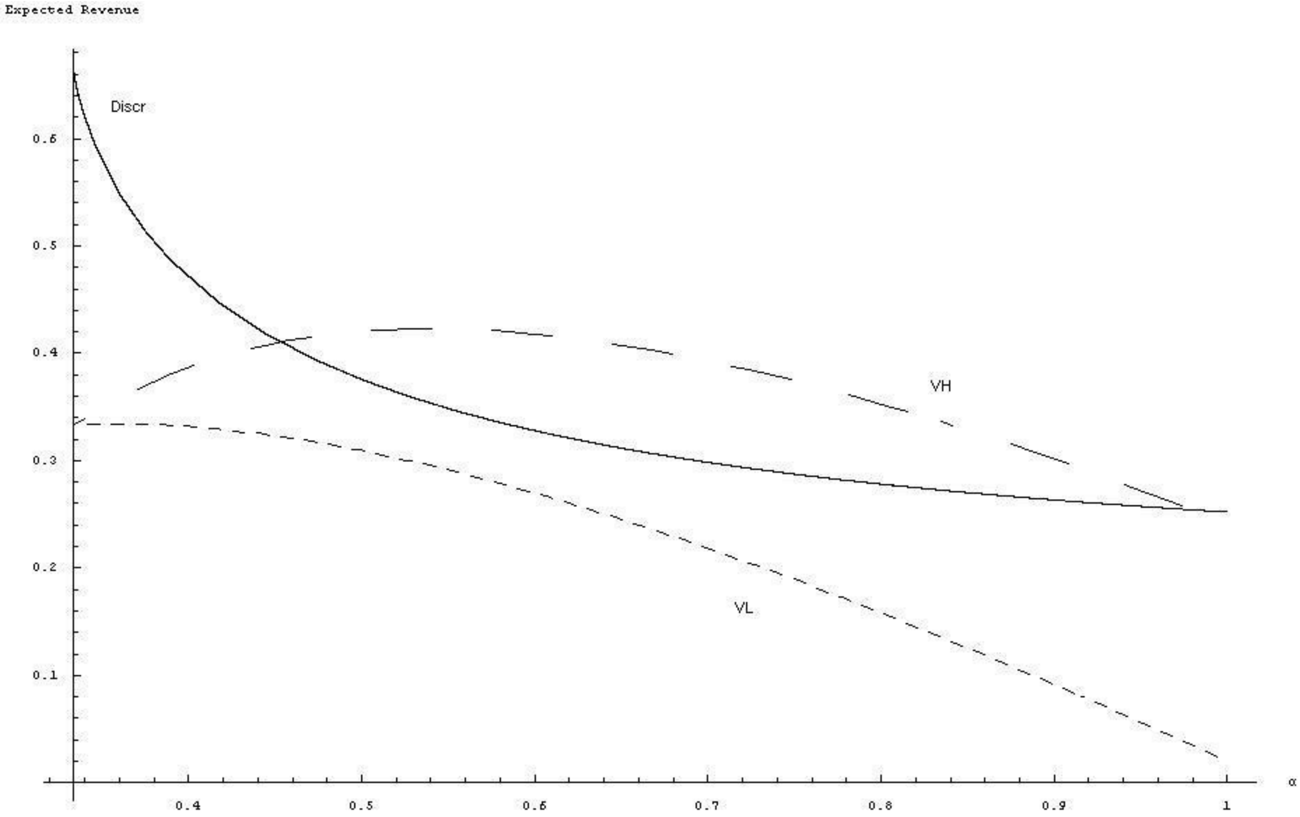
and we conclude that $\beta(\theta) = \alpha(1 - \frac{F(\tilde{\theta})}{F(\theta)})$ whenever $\tilde{\theta} \leq \theta \leq 1$.

Suppose that $0 \leq \theta \leq \tilde{\theta}$. Since $\psi'(b|\theta) \leq 0$ for each $b \in [0, \alpha(1 - F(\tilde{\theta}))]$, it follows that

$$\psi(0|\theta) = \max_{0 \leq b \leq \alpha(1 - F(\tilde{\theta}))} \psi(b|\theta) \geq \psi(\alpha(1 - F(\tilde{\theta}))|\theta) = \max_{b \geq \alpha(1 - F(\tilde{\theta}))} \psi(b|\theta)$$

and we conclude that $\beta(\theta) = 0$ whenever $0 \leq \theta \leq \tilde{\theta}$. □

Figure 1:



Revenue Comparison: V_H-Higher bound Vickrey, V_L-Lower bound Vickrey, Discr-Discriminatory