Lecture #12:

0.0.1  Graph Algorithms: Maximum Flow (Chapter 26)

Given a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a capacity $c(u, v) \geq 0$. If $(u, v) \notin E$ we assume that $c(u, v) = 0$. We also distinguish two vertices $s$ (origin = source) and $t$ (destination = sink). A flow in $G$ is a function $f : V \times V \rightarrow \mathbb{R}$ that satisfies the following three properties:

- **Capacity Constraint**: $f(u, v) \leq c(u, v) \forall u, v \in V$
- **Skew symmetry**: $f(u, v) = -f(v, u) \forall u, v \in V$
- **Flow conservation**: $\sum_{v \in V} f(u, v) = 0 \forall u \in V - \{s, t\}$

$f(u, v)$ is called the net flow from $u$ to $v$. The value of $f$ denoted by $|f|$ is equal to $\sum_{v \in V} f(s, v)$.

In the **maximum flow problem**, we want the flow $f$ with maximum $|f|$. For example, consider the following problem:

\[
\begin{align*}
\text{The value of this flow is 19.}
\end{align*}
\]
Cancellation: Consider the following diagram:

In the first diagram, we have the capacities; the second has a flow of 8 from $a$ to $b$; the third has an additional flow of 3 from $b$ to $a$; cancelling the flow in both directions gives us the next diagram with 5 from $a$ to $b$, and none in the other direction. Finally, an additional flow of 7 from $b$ to $a$, cancels the previous flow and introduces a flow of 2 from $b$ to $a$.

Because of this, we define the concept of **residual capacity** as follows: $c_f(u, v) = c(u, v) - f(u, v)$. The residual capacity for the above case would look like:

And a residual network as follows: Given a flow network $G = (V, E)$, $c : E \rightarrow \mathbb{R}^+$, and a flow $f$, the residual network $G_f = (V, E_f)$ where $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$. **Please note that $E_f$ may contain edges not present in $E$.** The main step of the algorithm is to find a directed path from $s$ to $t$ in $G_f$ called an **augmenting path** and saturate it. To saturate a path $P$, we add a flow equal to $\min_{(u, v) \in P} c_f(u, v)$ along the path. This step is repeated until there is no such path. **Please note that after any flow change,**
you must redefine the residual network. We show all this by an example below:

Residual networks are on the left column and flows on the right. The last step does not have a path in the residual network. The solution at this time is optimal.

The path is the residual network is found using BFS and BFS is shown
below:

Q is empty