Lecture #12:

0.0.1 Graph Algorithms: Maximum Flow (Chapter 26)

Given a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a **capacity** $c(u, v) \geq 0$. If $(u, v) \notin E$ we assume that $c(u, v) = 0$. We also distinguish two vertices s (origin = source) and t (destination = sink). A **flow** in G is a function $f : V \times V \longrightarrow \mathbf{R}$ that satisfies the following three properties:

Capacity Constraint: $f(u, v) \leq c(u, v) \forall u, v \in V$

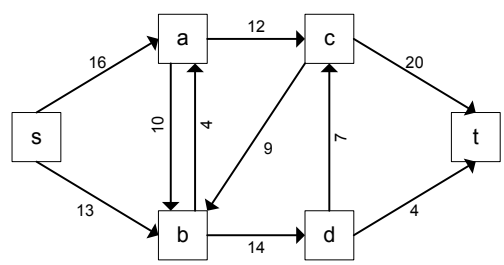
Skew symmetry: $f(v, u) = -f(u, v) \forall u, v \in V$

Flow conservation: $\sum_{v \in V} f(u, v) = 0 \forall u \in V - \{s, t\}$

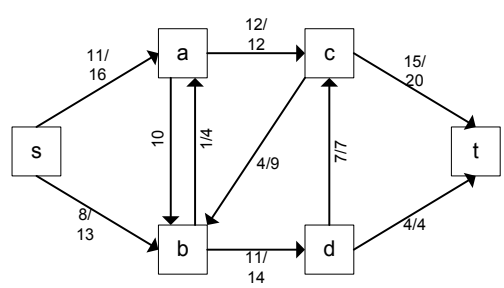
$f(u, v)$ is called the net flow from u to v. The value of f denoted by $|f|$ is equal to $\sum_{v \in V} f(s, v)$.

In the **maximum flow problem**, we want the flow f with maximum $|f|$.

For example, consider the following problem:

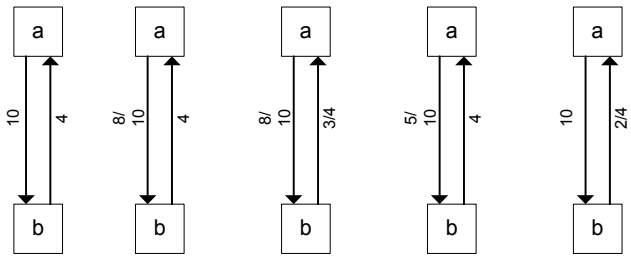


The following is a flow:



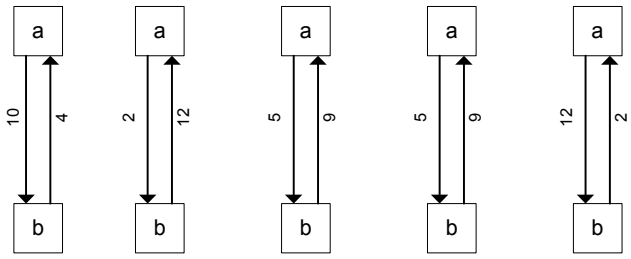
The value of this flow is 19.

Cancellation: Consider the following diagram:



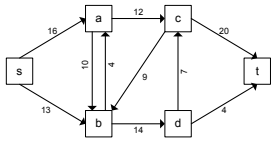
In the first diagram, we have the capacities; the second has a flow of 8 from a to b; the third has an additional flow of 3 from b to a; cancelling the flow in both directions gives us the next diagram with 5 from a to b, and none in the other direction. Finally, an additional flow of 7 from b to a, cancels the previous flow and introduces a flow of 2 from b to a.

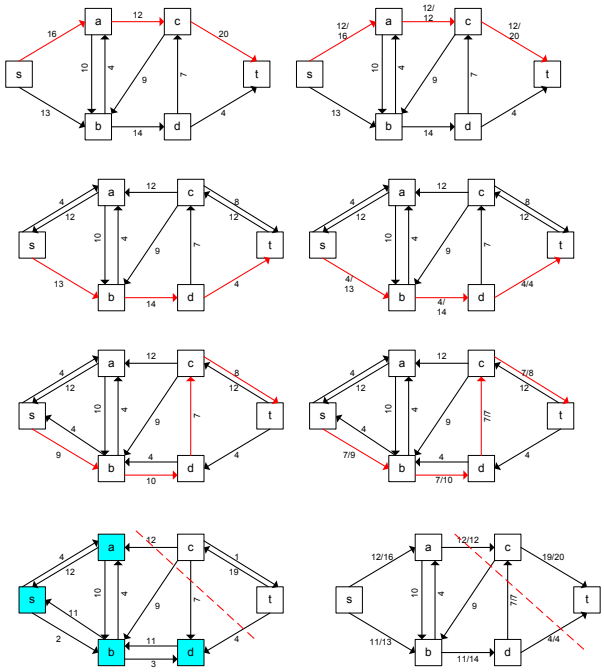
Because of this, we define the concept of **residual capacity** as follows: $c_f(u,v) = c(u,v) - f(u,v)$. The residual capacity for the above case would look like:



And a residual network as follows: Given a flow network $G = (V, E)$, $c : E \longrightarrow \mathbf{R}_+$, and a flow f , the residual network $G_f = (V, E_f)$ where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$. **Please note that E_f may contain edges not present in E.** The main step of the algorithm is to find a directed path from s to t in G_f called an **augmenting path** and **saturate** it. To saturate a path P, we add a flow equal to $\min_{(u,v)\in P}[c_f(u,v)]$ along the path. This step is repeated until there is no such path. Please note that after any flow change,

you must redefine the residual network. We show all this by an example below:





Residual networks are on the left column and flows on the right. The last step does not have a path in the residual network. The solution at this time is optimal.

The path is the residual network is found using BFS and BFS is shown

below: