Lecture #2:

Asymptotic Notation We study the notations $\Theta, O, \Omega, o, \omega$ in this section. All definitions used here are in the book in Chapter 2. We will give examples of some of them.

Definition 1 $\Theta(g(n)) = \{f(n) : \exists$ *constants* $c_1, c_2,$ *and* n_0 *such that* $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \forall n \geq n_0\}$

Definition 2 $O(g(n)) = \{f(n) : \exists$ *constants* $c_2,$ *and* n_0 *such that* $0 \leq f(n) \leq c_2 g(n) \forall n \geq n_0\}$

Definition 3 $\Omega(g(n)) = \{f(n) : \exists$ *constants* $c_1,$*and* n_0 *such that* $0 \leq c_1 g(n) \leq f(n) \forall n \geq n_0\}$

Definition 4 $o(g(n)) = \{f(n) :$ *for any* $c > 0, \exists$ *constant* $n_0(c)$ *such that* $0 \leq f(n) \leq c g(n) \forall n \geq n_0(c)\}$

One easy way to check this is to check that $\lim_{n\to\infty}[\frac{f(n)}{g(n)}] = 0.$

Definition 5 $\omega(g(n)) = \{f(n) : \forall c > 0, \exists$ *constant* $n_0(c)$ *such that* $0 \leq c g(n) \leq f(n) \forall n \geq n_0(c)\}$

Examples

- $f(n) = an^2 + bn + c; a > 0; g(n) = n^2$. Then $f(n) = \Theta(g(n))$. Only the leading power counts.
- $10^6 n = o(n^2)$. Smaller powers grow slower than larger powers.
- If $f(n) = o(g(n))$ then it follows that $f(n) = O(g(n))$ automatically.
- $10^{-6} n^2 = \omega(n)$
- If $f(n) = O(g(n)) \iff g(n) = \Omega(f(n)$
- If $f(n) = o(g(n)) \iff g(n) = \omega(f(n)$
- If $f(n) = n$ and $g(n) = n^{1+\sin n}$ then we can not say much in comparing the functions asymptotically.
- $n! = o(n^n)$
- If $f(n) = n^b$ and $g(n) = a^n$ with $a > 1$, then $f(n) = o(g(n))$. This says that any polynomially growing function is smaller than an exponentially growing function asymptotically. This is an important result and so we will show its proof.

  Let $h(n) = \frac{f(n)}{g(n)} = \frac{n^b}{a^n}$. We show that $\lim_{n\to\infty} h(n) = 0$ by showing that $\frac{h(n+1)}{h(n)} < 1$ for $n \geq n_1$.

  $\frac{h(n+1)}{h(n)} = (\frac{n+1}{n})^b \frac{1}{a}$ and hence the result follows.

- If $f(n) = \lg^b n = (\lg n)^b$ and $g(n) = n^a$ with $a > 0$, then $f(n) = o(g(n))$. This says that any power of logarithmically growing function is smaller than a polynomially growing function asymptotically. In particular, $\lg n = o(n^\epsilon)$ for all $\epsilon > 0$.

  By the previous result, we know that $\lim_{m\rightarrow\infty}[\frac{m^b}{c^m}] = 0$ when $c > 1$. Let $c = 2^a, a > 0$ and $m = \lg n$. $c^m = (2^a)^{\lg n} = 2^{a \lg n} = (2^{\lg n})^a = n^a$ Hence this result follows.

- $\lg[n!] = \sum_{i=1}^n \lg i = \Theta(n \lg n)$.

  $\sum_{i=1}^n \lg i \le \sum_{i=1}^n \lg n = n \lg n$. Hence we may choose $c_2 = n_0 = 1$ in showing $\sum_{i=1}^n \lg i = O(n \lg n)$.

  $\sum_{i=1}^n \lg i \ge \sum_{i=\lceil\frac{n}{2}\rceil}^n \lg i \ge \sum_{i=\lceil\frac{n}{2}\rceil}^n \lg\frac{n}{2} = \lfloor\frac{n}{2}\rfloor \lg\frac{n}{2} \ge \frac{n}{2}\lg\frac{n}{2} - \lg\frac{n}{2} = \frac{n}{2}\lg n - \frac{n}{2} - \lg n + 1 = (n\lg n)[\frac{1}{2} - \frac{\frac{n}{2} - \lg n + 1}{n \lg n}]$

  $\lim_{n\rightarrow\infty}[\frac{1}{2} - \frac{\frac{n}{2} - \lg n + 1}{n \lg n}] = \frac{1}{2}$ Hence $[\frac{1}{2} - \frac{\frac{n}{2} - \lg n + 1}{n \lg n}] > \frac{1}{4}$ for sufficiently large n. Hence $\sum_{i=1}^n \lg i = \Omega(n \lg n)$.

  These two together give what we want.

The last three results are very important.

Some Relations: $[f(n) = O(g(n))] \Rightarrow [g(n) = \Omega(f(n))]$
$[f(n) = \Theta(g(n))] \Leftrightarrow [g(n) = \Theta(f(n))]$
$[f(n) = \Theta(g(n)) \Longleftrightarrow [\{f(n) = O(f(n))\}$ **and** $\{f(n) = \Omega(g(n))\}]$
$[f(n) = o(g(n))] \Leftrightarrow [\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = 0\ ]$
$[f(n) = o(g(n))] \Rightarrow [f(n) = O(g(n))]$
$[f(n) = o(g(n)) \Leftrightarrow [g(n) = \omega(f(n))]$
Hence $[f(n) = \omega(g(n))] \Rightarrow [g(n) = o(f(n))] \Rightarrow [g(n) = O(f(n))] \Rightarrow [f(n) = \Omega(g(n))]$

$[\lim_{n\longrightarrow\infty} \frac{f(n)}{g(n)} = c$ where $0 < c < \infty] \Longrightarrow f(n) = \Theta(g(n))$ but the converse is not true.