Bimatrix Games

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1 Bimatrix Games

These are **two person non-zero(constant)-sum games** in which each player has **finitely many pure strategies**. We call these non-zero(constant)-sum games because the interests of the players is not required to be exactly opposed to each other. Of course, these include zero(constant)-sum games and are a true generalization of zero(constant)-sum games. But the methods used to analyze them are different. They also bring our many more difficulties as shown in the initial part. Suppose player 1 has m pure strategies and player 2 has n pure strategies. Let A be an $m \times n$ matrix representing payoffs to player 1and similarly let B be an $m \times n$ matrix representing the payoff matrix to player 2. If $A+B=0(\alpha J)$ where the right side is an $m \times n$ matrix all of whose entries are zero(J is a matrix all of whose entries are equal to 1 and α is a number), then we have the zero(constant)-sum case. Else we have the non-zero(constant)-sum case. A pair of (mixed) strategies $x^* \in X = \{x : x \geq 0; \sum_{i=1}^{m} x_i = 1\}$ and $y^* \in Y = \{y : y \geq 0; \sum_{j=1}^{n} y_j = 1\}$ which satisfy the relations:

$$\begin{aligned}(x^*)^t A y^* &\geq x^t A y^* \qquad \forall x \in X \\ (x^*)^t B y^* &\geq (x^*)^t B y \qquad \forall y \in Y\end{aligned}$$

is called a Nash equilibrium. J. Nash showed that these exist as in the following theorem. His proof uses the well known Brower fixed point theorem. [There are other fixed point theorems that are more powerful and can extend these results]

Theorem 1 *(Brower) Let $T : S \to S$ be a continuous function that maps a compact convex set $S \subseteq R^n$ to itself. Then there is a point $x \in S$ such that $T(x) = x$. Such a point is called a fixed point for T.*

Theorem 2 *Every bimatrix game has at least one equilibrium point.*
Proof. *(Owen, Luce and Raiffa): Let x and y be any pair of mixed strategies for the bimatrix game (A, B). Define:*

$$\begin{aligned} c_i &= \max[A_{i,.}y - x^tAy, 0]; \qquad 1 \le i \le m \\ d_j &= \max[x^tB_{.,j} - x^tBy, 0]; \qquad 1 \le j \le n \\ x'_i &= \frac{x_i + c_i}{1 + \sum_{k=1}^m c_k} \qquad 1 \le i \le m \\ y'_j &= \frac{y_j + d_j}{1 + \sum_{l=1}^n d_l} \qquad 1 \le j \le n \end{aligned}$$

Think of the transformation T by the relation:

$$T(x, y) = (x', y')$$

Such a T is clearly continuous and maps $X \times Y$ to itself [show this]. If (x, y) is an equilibrium pair of strategies, $x^tAy \ge (e_i)^tAy = A_{i,.}y$ and so $c_i = 0$ for all i. [Here e_i denotes the unit vector with a 1 in the i^{th} position.] Similarly, $d_j = 0$ for all j. Hence, $(x', y') = (x, y)$ for such a pair. Hence equilibrium pairs of strategies are fixed points of this transformation. Now we show the converse is also true. ■
Proof. *Since both X and Y are compact convex sets so is the Cartesian product. Hence we can apply Brower fixed point theorem and there is fixed point. Let us denote the fixed point by (x^*, y^*). Now we use the relations that $T(x^*, y^*) = (x^*, y^*)$ to show that these are an equilibrium pair of strategies for the game.*

$$\begin{aligned} x_i^* &= \frac{x_i^* + c_i^*}{1 + \sum_{k=1}^m c_k^*} \qquad 1 \le i \le m \\ y_j^* &= \frac{y_j^* + d_j^*}{1 + \sum_{l=1}^n d_l^*} \qquad 1 \le j \le n \end{aligned}$$

where

$$\begin{aligned} c_i^* &= \max[A_{i,.}y^* - (x^*)^tAy^*, 0]; \qquad 1 \le i \le m \\ d_j^* &= \max[x^tB_{.,j} - (x^*)^tBy^*, 0]; \qquad 1 \le j \le n \end{aligned}$$

■

Claim 3 *If (x^*, y^*) is not an equilibrium pair of strategies, at least one of the values of c_i^* or one of the values of d_j^* is strictly positive.*
Proof. *If the claim is not true, then it follows that*

$$(x^*)^t A y^* \geq A_{i,.} y^* = (e_i)^t A y^* \qquad \text{for } 1 \leq i \leq m$$

Multiplying the the i^{th} of these inequalities by x_i^* and adding [this is permitted since $x_i^* \geq 0$], we get

$$(x^*)^t A y^* \geq x^t A y^* \qquad \forall x \in X$$

Similarly we can show that

$$(x^*)^t B y^* \geq (x^*)^t A y \qquad \forall y \in Y$$

■

Proof. *(of the main theorem continued): Suppose without loss some $c_i^* > 0$ (equivalently $A_{i,.} y^* > (x^*)^t A y^*$). Hence $\sum_{i=1}^m c_i^* > 0$. But $(x^*)^t A y^*$ is weighted average of $\{A_{i,.} y^*\}_{i=1}^m$. Hence, there is some other index p such that*

$$A_{p,.} y^* < (x^*)^t A y^*$$
$$x_p^* > 0$$

So, $c_p^* = 0$ and hence

$$x_p' = \frac{x_p^*}{1 + \sum_{k=1}^m c_k^*} < x_p^*$$

and therefore $x' \neq x^*$. If instead some $d_j^* > 0$, we will show that $y' \neq y^*$. In either case this will not be a fixed point of this transformation. Hence fixed points are equilibrium pairs and therefore equilibrium pairs of strategies exist for any bimatrix game. Proof of existence of a Nash equilibrium for n-person games is similar. ■

There is alternate proof based on Kakutani's fixed point theorem. This is based on the fact that in these games, the set of best response for player i against a strategy profile of the remaining players is a compact convex set (it is also polyhedral if the set of pure strategies of each player is finite). This is exercise 2.4 in your homework assignments.

Here is the proof:

Proof. Let $(p_1, p_2, ..., p_n)$ be a mixed strategy profile. [Note that each of these vectors p_i is finite dimensional.]Let $B_i(p_1, p_2, ..., p_n)$ be the set of "best responses" for player i against p_{-i}. Recall from HW 2.4, that these sets B_i are compact convex sets. Hence $B_1 \times B_2 \times ... \times B_n$ is also a compact convex set. Consider the correspondence

$$(p_1, p_2, ..., p_n) \to B_1 \times B_2 \times ... \times B_n$$

It maps a strategy profile to a compact convex subset of strategy profiles. It is also upper-hemi-continuous (see definition below). Hence by Kakutani's theorem, $\exists$ a strategy profile $(p_1, p_2, ..., p_n)$ which is a fixed point of this correspondence. This clearly is a Nash equilibrium. ■

Definition 4 *A correspondence* $\Gamma : A \to B$ *is said to be upper hemicontinous at the point* $x \in A$ *if for any open neighborhood* V *of* $\Gamma(x)$*, there exists a neighborhood* U *of* x *such that* $\Gamma(y)$ *is a subset of* V *for all* $y \in U$*.*

Definition 5 *The graph of a correspondence* $\Gamma : A \to B$ *is the set* $\{(x, y) : y \in \Gamma(x); x \in A, y \in B\}$*. This graph is said to be closed if the set is closed. If the sets* A*,* B *are compact and* $\Gamma(x)$ *is closed for each* x *this concept is equivalent to upper hemicontinuity. It is this that is used to assure that the conditions of kakutani's theorem are valid for our case.*

Pairs of strategies (x^*, y^*)that solve

$$\min(x^*)^t Ay = \max_{x \in X} \min_{y} x^t Ay$$
$$\max xAy^* = \max_{y \in Y} \min_{x \in X} x^t Ay$$

do not necessarily form Nash equilibrium pairs for a bimatrix game.

The computational problem of finding these equilibrium pairs for a bimatrix game was "solved" by formulating it as a special case of the well known "Linear Complementarity Problem" proposed by Lemke and Howson. The general LCP has many other applications (see book by R.W. Cottle, J.S. Pang and R.E. Stone). Lemke- Howson algorithm is a finite algorithm but it is not guaranteed to be polynomially bounded – indeed no such algorithm is known.

1.0.1 Linear Complementarity Problem (LCP) & Bimatrix Games

Given a matrix $M \in R^{n \times n}$, a vector $q \in R^n$, the linear complementarity problem is to find vectors $w, z \in R^n$ satisfying the relations:

$$\begin{aligned} w &= Mz + q \\ w &\geq 0; z \geq 0 \\ w^t z &= 0 \end{aligned}$$

Please note that $[w \geq 0; z \geq 0; w^t z = 0] \Leftrightarrow [w_i > 0 \Rightarrow z_i = 0; z_i > 0 \Rightarrow w_i = 0]$; hence, at most one of each pair $\{w_i, z_i\}$ can be nonzero. This is the reason for the term "complementary" in the name of the problem. For this formulation, it is more convenient to think of the payoff matrices as "loss" matrices. Let $[a_{i,j}, b_{i,j}]$ represent the loss to player 1 and 2 respectively if player 1 uses pure strategy i and player 2 uses pure strategy j. Then, a pair of mixed strategies (x^*, y^*) are a Nash equilibrium pair if they satisfy the relations:

$$\begin{aligned} (x^*)^t A y^* &\leq x^t A y^* \qquad \forall x \in X \\ (x^*)^t B y^* &\leq (x^*)^t B y \qquad \forall y \in Y \end{aligned}$$

This is equivalent to the system:

$$\begin{aligned} [(x^*)^t A y^*] e_m &\leq A y^* \\ [(x^*)^t B y^*] e_n &\leq B^t x^* \end{aligned}$$

where e_m and e_n refer to vector of all 1's of size m and n respectively. We can show that this reduces to the linear complementarity problem:

$$\begin{aligned} u &= Ay - e_m \\ v &= B^t x - e_n \\ x &\geq 0; y \geq 0; u \geq 0; v \geq 0 \\ x^t u &= 0; y^t v = 0 \end{aligned}$$

or equivalently the problem:

$$\begin{aligned} w &= Mz + q \\ w &\geq 0; z \geq 0 \\ w^t z &= 0 \end{aligned}$$

with

$$M = \begin{array}{|c|c|}\hline 0 & A \\ \hline B^t & 0 \\ \hline \end{array} \quad ; q = \begin{array}{|c|}\hline -e_m \\ \hline -e_n \\ \hline \end{array}$$
$$w = \begin{array}{|c|}\hline u \\ \hline v \\ \hline \end{array} \quad ; z = \begin{array}{|c|}\hline x \\ \hline y \\ \hline \end{array}$$

To show the equivalence, suppose (x^*, y^*) is an equilibrium pair of strategies. So clearly, $x^* \geq 0; y^* \geq 0$. Suppose (and we will show that this is without loss of generality) that $A > 0; B > 0$. Let

$$\begin{aligned} x &= \frac{x^*}{(x^*)^t B y^*} \\ y &= \frac{y^*}{(x^*)^t A y^*} \\ u &= Ay - e_m \\ v &= B^t x = e_n \end{aligned}$$

Since $\sum_{i=1}^m x_i^* = 1$ and $\sum_{j=1}^n y_j^* = 1$, it follows that $(x^*)^t B y^* > 0; (x^*)^t A y^* > 0$ by our assumption that $A > 0, B > 0$. Thus, $x \geq 0; y \geq 0$. $[[(x^*)^t A y^*] e_m \leq A y^*] \Rightarrow [Ay \geq e_m]$ and $[[(x^*)^t B y^*] e_n \leq B^t x^*] \Rightarrow [B^t x \geq e_n]$. Hence $u \geq 0; v \geq 0$.

$$\begin{aligned} x^t u &= \frac{(x^*)^t [A \frac{y^*}{(x^*)^t A y^*} - e_m]}{(x^*)^t B y^*} \\ &= \frac{1}{(x^*)^t B y^*} - \frac{1}{(x^*)^t B y^*} \\ &= 0 \end{aligned}$$

Similarly $y^t v = 0$. Hence, (x, y, u, v) is a solution to the LCP.

To do the converse suppose x^0, y^0, u^0, v^0 solves the linear complementarity problem. Clearly, neither x^0 nor y^0 can be equal to the 0 vector since u^0, v^0 are nonnegative. Hence $\sum_{i=1}^m x_i^0 > 0; \sum_{j=1}^n y_j^0 > 0$. Now let

$$\begin{aligned} x^* &= \frac{x^0}{\sum_{i=1}^m x_i^0} \\ y^* &= \frac{y^0}{\sum_{j=1}^n y_j^0} \end{aligned}$$

These are mixed strategies for the two players. Now we show that (x^*, y^*) is a Nash equilibrium pair. To do this, we need to show that

$$[(x^*)^t Ay^*]e_m \leq Ay^*$$

which is equivalent to showing;

$$Ay^* = A[\frac{y^0}{(e_n)^t y^0}] \geq [(x^*)^t Ay^*]e_m = \frac{(x^0)^t Ay^0}{[(e_m)^t x^0] \bullet [(e_n)^t y^0]}$$

Since (x^0, y^0, u^0, v^0) solves the LCP,

$$0 = (x^0)^t u^0 = (x^0)^t [Ay^0 - e_m]$$

and so

$$(e_m)^t x^0 = (x^0)^t Ay^0$$

Similarly,

$$(e_n)^t y^0 = (x^0)^t By^0$$

$$Ay^* = A[\frac{y^0}{(e_n)^t y^0}] \geq [(x^*)^t Ay^*]e_m = \frac{(x^0)^t Ay^0}{[(e_m)^t x^0] \bullet [(e_n)^t y^0]}$$

Since $u^0 = Ay^0 - e_m \geq 0$, it follows that $Ay^0 \geq e_m$. Therefore,

$$Ay^* = A\frac{y^0}{(e_n)^t y^0} \geq [\frac{(x^0)^t Ay^0}{(e_m)^t x^0}][\frac{1}{(e_n)^t y^0}]e_m = [(x^*)^t Ay^*]e_m$$

which is what we set out to prove. Similarly we can show that

$$B^t x^* \geq [(x^*)^t By^*]e_n$$

Thus we have shown that (x^*, y^*) is an equilibrium pair of strategies.

Lemma 6 *Let A and B be "loss" matrices in a bimatrix game. Let k be a scalar and let J be a matrix of all 1's of the same size as A and B. Let $C = A + kJ; D = B + kJ$. Any equilibrium pair of strategies for the pair (A, B) is also an equilibrium pair of strategies for the pair (C, D).*

Proof of this lemma is quite straightforward and left to the reader. If k is a sufficiently large positive value, then matrices C and D will be positive matrices and it is in this sense that this assumption is without loss of generality. It will be a digression to discuss the algorithm of Lemke-Howson here; suffice it to say that it is a finite procedure but not guaranteed to be polynomially bounded.

1.1 Papadimitriou-Daskalakis Formulation:

Here we continue to think of matrices A and B as "gain" or payoff matrices as usual in Game Theory.

Then, a pair of mixed strategies (x^*, y^*) are a Nash equilibrium pair if they satisfy the relations:

$$\begin{aligned} (x^*)^t A y^* &\geq x^t A y^* \qquad \forall x \in X \\ (x^*)^t B y^* &\geq (x^*)^t B y \qquad \forall y \in Y \end{aligned}$$

This is equivalent to the system:

$$\begin{aligned} [(x^*)^t A y^*] e_m &\geq A y^* \\ [(x^*)^t B y^*] e_n &\geq B^t x^* \end{aligned}$$

where e_m and e_n refer to vector of all 1's of size m and n respectively. We can show that this reduces to the "modified linear complementarity problem":

$$\begin{aligned} u + Ay &= e_m \\ v + B^t x &= e_n \\ x &\geq 0; y \geq 0; u \geq 0; v \geq 0 \\ x^t u &= 0; y^t v = 0 \end{aligned}$$

or equivalently the problem:

$$\begin{aligned} w + Mz &= q \\ w &\geq 0; z \geq 0 \\ w^t z &= 0 \end{aligned}$$

with

$$M = \begin{array}{|c|c|} \hline 0 & A \\ \hline B^t & 0 \\ \hline \end{array} \qquad ; q = \begin{array}{|c|} \hline e_m \\ \hline e_n \\ \hline \end{array}$$

$$w = \begin{array}{|c|} \hline u \\ \hline v \\ \hline \end{array} \qquad ; z = \begin{array}{|c|} \hline x \\ \hline y \\ \hline \end{array}$$

To show the equivalence, suppose (x^*, y^*) is an equilibrium pair of strategies. So clearly, $x^* \gneqq 0; y^* \gneqq 0$. Suppose without loss of generality that $A >$

$0; B > 0$. Let

$$
\begin{aligned}
x &= \frac{x^*}{(x^*)^t B y^*} \gneqq 0 \\
y &= \frac{y^*}{(x^*)^t A y^*} \gneqq 0 \\
u &= -Ay + e_m \geq 0 \\
v &= -B^t x + e_n \geq 0
\end{aligned}
$$

Since $\sum_{i=1}^m x_i^* = 1$ and $\sum_{j=1}^n y_j^* = 1$, it follows that $(x^*)^t B y^* > 0; (x^*)^t A y^* > 0$ by our assumption that $A > 0, B > 0$. Thus, $x \gneqq 0; y \gneqq 0$. $[[(x^*)^t A y^*] e_m \geq A y^*] \Rightarrow [Ay \leq e_m]$ and $[[(x^*)^t B y^*] e_n \geq B^t x^*] \Rightarrow [B^t x \leq e_n]$. Hence $u \geq 0; v \geq 0$.

$$
\begin{aligned}
x^t u &= \frac{(x^*)^t [-A \frac{y^*}{(x^*)^t A y^*} + e_m]}{(x^*)^t B y^*} \\
&= -\frac{1}{(x^*)^t B y^*} + \frac{1}{(x^*)^t B y^*} \\
&= 0
\end{aligned}
$$

Similarly $y^t v = 0$. Hence, $(x \gneqq 0, y \gneqq 0, u \geq 0, v \geq 0)$ is a solution to the LCP.

To do the converse suppose $x^0 \gneqq 0, y^0 \gneqq 0, u^0 \geq 0, v^0 \geq 0$ solves the linear complementarity problem. Hence $\sum_{i=1}^m x_i^0 > 0; \sum_{j=1}^n y_j^0 > 0$. Now let

$$
\begin{aligned}
x^* &= \frac{x^0}{\sum_{i=1}^m x_i^0} \\
y^* &= \frac{y^0}{\sum_{j=1}^n y_j^0}
\end{aligned}
$$

These are mixed strategies for the two players. Now we show that (x^*, y^*) is a Nash equilibrium pair. To do this, we need to show that

$$
[(x^*)^t A y^*] e_m \geq A y^*
$$

which is equivalent to showing;

$$
A y^* = A[\frac{y^0}{(e_n)^t y^0}] \geq [(x^*)^t A y^*] e_m = \frac{(x^0)^t A y^0}{[(e_m)^t x^0] \bullet [(e_n)^t y^0]}
$$

Since (x^0, y^0, u^0, v^0) solves the LCP,

$$0 = (x^0)^t u^0 = (x^0)^t[-Ay^0 + e_m]$$

and so

$$(e_m)^t x^0 = (x^0)^t Ay^0$$

Similarly,

$$(e_n)^t y^0 = (x^0)^t By^0$$

$$Ay^* = A[\frac{y^0}{(e_n)^t y^0}] \leq [(x^*)^t Ay^*]e_m = \frac{(x^0)^t Ay^0}{[(e_m)^t x^0] \bullet [(e_n)^t y^0]}$$

Since $u^0 = Ay^0 - e_m \geq 0$, it follows that $Ay^0 \geq e_m$. Therefore,

$$Ay^* = A\frac{y^0}{(e_n)^t y^0} \leq [\frac{(x^0)^t Ay^0}{(e_m)^t x^0}][\frac{1}{(e_n)^t y^0}]e_m = [(x^*)^t Ay^*]e_m$$

which is what we set out to prove. Similarly we can show that

$$B^t x^* \leq [(x^*)^t By^*]e_n$$

Thus we have shown that (x^*, y^*) is an equilibrium pair of strategies.

1.2 Symmetric Games and Symmetrization

An n-player game is said to be symmetric if each player's set $S = \{1, 2, ..., s\}$ of pure strategies is the same and payoff function is the same and equal to $\pi(\sigma; n_1, n_2, ..., n_s)$ where n_i is the number of players using pure strategy i.

Definition 7 *A **symmetric equilibrium** for a symmetric game is a Nash equilibrium where all players use the same strategy. .*

Theorem 8 *Nash: There exists a symmetric equilibrium for a symmetric game.*

Proof. *In using the fixed point theorem use the set which is the intersection of product of the sets for each player intersected with the set that requires the choice of strategies across players to be the same.* ■

Theorem 9 *Every two person game is equivalent to two person symmetric game.*

Proof. *Let the bimatrix game have payoff matrices A and B. Without loss of generality, we may assume that $A > 0$ and $B > 0$. Consider the symmetric game with matrices C and C^t where*

$$C = \begin{array}{|c|c|} \hline 0 & A \\ \hline B^t & 0 \\ \hline \end{array}$$

By theorem above, this game has a symmetric Nash equilibrium $z^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$

Claim 10 $x^* \gneqq 0; y^* \gneqq 0$

Proof. *Suppose $x^* = 0$ or $y^* = 0$. Consider Player I using strategy* $z = \begin{bmatrix} x \\ y \end{bmatrix}$ *with $x \gneqq 0; y \gneqq 0$ against Player II using $z^* = (x^*, y^*)$.*

$$\begin{aligned} (z^*)^t C z^* &= (x^*)^t A y^* + (x^*)^t B^t y^* = 0 \\ &< (x)^t A y^* + (x^*)^t B^t y \end{aligned}$$

The last of these follows from $A > 0, B > 0$ and $x \gneqq 0, y \gneqq 0$. This would lead to a contradiction that z^* is a Nash equilibrium. Hence the claim. ■

Proof. *(continued) Let*

$$\begin{aligned} \bar{x} &= \frac{x^*}{(e_m)^t x^*} \\ \bar{y} &= \frac{y^*}{(e_n)^t y^*} \\ z_1 &= \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}; z_2 = \begin{bmatrix} 0 \\ \bar{y} \end{bmatrix} \end{aligned}$$

Using the fact that z^* is the best response to z^*, we can obtain that z_1 is the best response to z_2 and hence $(\bar{x}, \bar{y})$ is a Nash equilibrium for the original game. ■

■

1.3 Computing Equilibrium in Two-person Symmetric Game[P,D]

Let M be the square matrix C in the above discussion; for this discussion, assume that size of this matrix is $n \times n$. Consider the system;

$$Mz \leq e_n$$
$$z \geq 0$$

Let $M_{i,\bullet}$ denote the i^{th} row of this matrix. Without loss, we may assume that elements of this matrix are nonnegative with no zero rows or zero columns. This implies that $P = \{z \in R^n : z \geq 0; Mz \leq e_n\}$ is nonempty and bounded and closed. P is said to be *nondegenerate* if at each vertex exactly n of these $2n$ inequalities are tight. By small perturbation, we can make any such polyhedron nondegenerate and we will assume this is the case from now on. Each vertex of this polytope is described by its tight constraints in the following manner: We use the index i as many times as the number of tight constraints among $\{M_{i,\bullet}z = 1; z_i = 0\}$. For example, if $M_{1,\bullet}z = 1$ and $z_1 = 0$ and $z_2 = 0$ but $M_{2,\bullet}z < 1$, this vertex will be denoted by $\{1, 1, 2\}$. So every vertex has n components. If a vertex has the label $\{1, 2, ..., n\}$ (i.e. all indices occur once), then one of the following is true: (a)$z = 0$; or (b)the normalized vector $\bar{z}$ is a symmetric Nash equilibrium.

To show the existence (and a computation mechanism) for a Nash equilibrium we do the following:

Select an index $a \in (1, 2, ..., n)$. Consider the set of all vertices whose index sets contain *all* indices from $\{1, 2, .., n\}$ *except possibly* the index a; call this set of vertices V. Each vertex in the polytope has exactly n neighbors obtained by relaxing one of the tight constraints. So for a vertex in V that contains all indices, there is only one neighbor in V; this obtained by relaxing one of the two constraints for the index a that is tight in this vertex.

Characterize other vertices in V by the index that is repeated say b. There are two constraints corresponding to index $b; M_{b,\bullet}z = 1$ and $z_b = 0$. Hence the vertex in V that has index b repeated has exactly two neighbors in V. The adjacency graph in V has degree 2 for vertices that do not have all indices and degree equal to 1 for those that have all indices represented. So if start at $z = 0$ which has all indices represented and move along this graph we must reach another vertex with all indices represented which must correspond to a symmetric Nash equilibrium! The process is finite because the size of V is finite. This is the crux of C.E. Lemke's algorithm.