Cooperative Game Theory

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Most of this follows Owen and Shubik and Wooldridge et al.

When we extend two person game theory to consider n person games for $n \geq 3$, there is little difference from non-cooperative game theory point of view. Existence of Nash equilibrium follows from similar arguments and all the difficulties we had with two person nonzero sum games show up here as well. But there is a new phenomenon here that must be taken into account:– that of *coalition* formation. Subsets of players could form a "cartel" and act in unison to gain more than they could if they acted independently. This forms one essential aspect of the game here. And this requires having binding contracts, using correlated mixed strategies, and transferable utility (so that the gain could be shared between the colluders in some way that all agree to). The main study here is to model the coalition formation, and gain sharing process. So we abstract away details and concentrate on important parts of the game.

Game Representation: Characteristic Function Forms Let $N = \{1, 2, ..., n\}$ be the set of players. Any nonempty subset S of N is called a *coalition*.

Definition 1 *By a **characteristic function** of an n-person game we mean a function v that assigns a value to each subset of players; i.e $v : 2^N \mapsto R$. We think of $v(S)$ as the payoff to the subset S of N if it acts in unison; some times it is also assumed that this is maximin payoff in that we also think all of $N - S$ act in unison (against S). $v(S)$ is called the value of the coalition S.*

When we go from games in extensive forms to normal forms, we abstract some details and only look at strategies to obtain a (mixed) equilibrium (for which we do not need the details that have been abstracted away). Similarly, in n person cooperative games where the study focuses on stable coalition formations, we abstract away even further and look only at the characteristic function form. It is implicitly assumed that a coalition S can distribute its value $v(S)$ to its members in any way they choose. Hence these are also called *transferable utility games* (TU games for short). How the distribution takes place is the main interest in these games. It is generally assume that $v(\{\phi\}) = 0; v(S) \geq 0 \ \forall S \subseteq N$.

Outcomes/Solutions An outcome of a game in characteristic form consists of:

(i)A partition of N into coalitions, called a *coalition structure*, and

(ii) a *payoff vector*, which distributes the value of each coalition to its members.

A coalition structure CS over N is a nonempty collection of nonempty subsets $CS = \{S_1, S_2, ..., S_k\}$ satisfying the relations:

$$\cup_{i=1}^{k} S_i = N; S_i \cap S_j = \phi \text{ if } i \neq j$$

The set of all coalition structures for a given set N of players is denoted by CS_N. $v(CS)$ denotes the sum $\sum_{j=1}^{k} v(S_j)$.

A vector $x = (x_1, x_2, ..., x_n)$ is payoff vector for a coalition structure $CS = \{S_1, S_2, ..., S_k\}$, over $N = \{1, 2, ..., n\}$ if

$$\begin{aligned} x_i \geq 0 & \qquad \forall i \in N \\ \sum_{i \in S_j} x_i \leq v(S_j) & \qquad 1 \leq j \leq k \end{aligned}$$

An outcome is a pair $[CS, x]$. $x(S) = \sum_{i \in S} x_i$ is called the payoff for the coalition S under x. x is said to be *efficient* in the outcome $[CS, x]$ if

$$\sum_{i \in S_j} x_i = v(S_j) \qquad 1 \leq j \leq k$$

A payoff vector x for a coalition structure CS_N is called an *imputation* if it is efficent and individually rational.

$$\begin{aligned} x_i \geq v(\{i\}) & \qquad \forall i \in N \\ \sum_{i \in S_j} x_i = v(S_j) & \qquad 1 \leq j \leq k \end{aligned}$$

The set of all imputations for a coalition structure $CS \in CS_N$ is denoted by $E(CS)$. If $CS = \{N\}$, then this is denoted by $E(N)$ or $E(v)$. If a payoff vector is an imputation, then each player prefers this to being alone. However, a group of players may want to deviate since it might be better for them and this would result in unstable conditions.

Subclasses of games in characteristic form:

Monotone Games: A game $[N, v]$ in characteristic form is *monotone* if

$$[S \subseteq T] \Rightarrow v(S) \leq v(T)$$

Most games are montone; nonmonotonicuty may arise because some players intesely dislike each other or becuase of the overhead charges for communication increase nonlinearly with size of the coaltion.

Superadditive Games: A game $[N, v]$ in characteristic form is said to be *superadditive* if

$$[S \cap T = \phi] \Rightarrow v(S \cup T) \geq v(S) + v(T)$$

It comes from the fact that S can assure itself $v(S)$ without help from any one and so also T can assure itself $v(T)$, then $S \cup T$ can assure itself the sum. Since we have assumed that characteristic function is nonnengative, it follows that superadditivity implies monotonicity. Most games are superadditive; indeed older books did not consider any others. Non-superadditive games arise from anit-trust or anti-monopoly regualtions.

In superadditive games, there is no compelling reason for players to form any coalition structure except $CS = \{N\}$ called the "grand" coalition. Hence the outcome for such a game is of the form $[N, x]$ where

$$\sum x_i = v(N)$$

A non-superaddtive game cna be transformed into a superadditive game by the following process: Let $T \subseteq N$ be any coalition. Let CS_T denote all coalition structures over T. Given a game $[N, v]$ we define a new game $[N^*, v^*]$ by

$$v^*(T) = \max_{CS \in CS_T} v(CS)$$

G^* is called the *superadditive cover* of the game G. $v^*(T)$ is the maximum that the players in set T cna achive by forming their own coalition structure in G.

Convex (Supermodualr) Games: A game is said to be *convex* or *supermodular* if

$$v(S \cup T) + v(s \cap T) \geq v(S) + v(T) \qquad \forall S, T \subseteq N$$

Theorem 2 *A game $G = [N, v]$ is convex iff*

$$[T \subset S; i \notin S] \Rightarrow [v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)]$$

A convex game is superadditive.

Definition 3 *A game v in characteristic function form is called a* ***constant sum game*** *if*

$$v(S) + v(N - S) = v(N) \qquad \forall S \subseteq N$$

It is clear from the super-additivity condition that the maximum the entire set of players can get is $v(N)$. Now we look into the questions of how to divide this total – it what does each player get – in a stable situation. Let $(x_1, x_2, ..., x_n)$ denote the payoff to the players. Clearly no player will accept less than what he can get for himself with no help from others. Hence one condition that this vector must satisfy (called individual rationality) is

$$x_i \geq v(\{i\}) \qquad \forall i$$

The second condition that is normally imposed (known as pareto-optimality) is to require

$$\sum_{i=1}^{n} x_i = v(N)$$

Any vector that satisfies these two conditions is called an *imputation*. The main question now is which of these in the set

$$E(v) = \{x : x_i \geq v(\{i\}); 1 \leq i \leq n; \sum_{i=1}^{n} x_i = v(N)\}$$

should the predicted outcome of this game be? The answer is easy in one case $v(N) = \sum_{i=1}^{n} v(\{i\})$ (this is the most uninteresting case!)

Definition 4 *A game is said to be* ***inessential*** *if* $v(N) = \sum_{i=1}^{n} v(\{i\})$.

By superadditivity, we have $v(N) \geq \sum_{i=1}^{n} v(\{i\})$. If equality holds, $E(v)$ contains only one point –

$$x_i = v(\{i\}) \qquad \forall i$$

Hence this the outcome of such games. From now on, we are interested only in essential games where $v(N) > \sum_{i=1}^{n} v(\{i\})$.

Definition 5 *Let $x, y \in E(v)$. We say that x dominates y via the coalition S [denoted by $x \succ_S y$] if*

$$x_i > y_i \qquad \forall i \in S$$
$$\sum_{i \in S} x_i \leq v(S)$$

Each player in S gets more under x than in y and the coalition S has enough to give its members the amount specified in x.

Definition 6 *We say x dominates y if the above is true for some S.*

If x dominates y then y is not stable. Games with same domination structure are in some sense equivalent and we make this precise by:

Definition 7 *Two n-person games u and v are said to be* ***isomorphic*** *if there is a function $f : E(u) \mapsto E(v)$ such that*

$$[x, y \in E(u); x \succ_S y] \Leftrightarrow [f(x) \succ_S f(y)]$$

We are preserving the domination structure.

Definition 8 *Two $n-$person games u and v are $S-$***equivalent*** *if there exists numbers $(a_1, a_2, ..., a_n)$ and $\beta > 0$ such that*

$$v(S) = \beta u(S) + \sum_{i \in S} a_i \qquad \forall S \subseteq N$$

Theorem 9 *If u and v are S-equivalent, then they are isomorphic. The converse is true for all constant sum games.*

Proof. Use the function $f(x) = \beta x + a$. ■

Since S-equivalence is indeed an equivalence relations, it is sufficient to study one member of each of its equivalence classes. Such representatives are called *normalized* games.

Definition 10 *An essential (characteristic function) game is said to be* $(0,1)$-*normalized if*

$$\begin{aligned} v(\{i\}) &= 0 \qquad \forall i \\ v(N) &= 1 \end{aligned}$$

Lemma 11 *A game is $S-$equivalent to exactly one game in* $(0,1)$ *normalized form.*

Another normalization used in the literature is the $(-1,0)$ normalization where

$$\begin{aligned} v(\{i\}) &= -1 \qquad \forall i \\ v(N) &= 0 \end{aligned}$$

We use the $(0,1)$ normalization. Thus, the set of all $(0,1)$ normalized games consist of $v \in 2^N$ that satisfy

$$\begin{aligned} v(\phi) &= 0 \\ v(\{i\}) &= 0 \qquad \forall i \\ v(N) &= 1 \\ [S \cap T = \phi] &\Rightarrow v(S \cup T) \geq v(S) + v(T) \end{aligned}$$

If the game is also a constant sum game it satisfies the relation

$$v(S) + v(N - S) = v(N)$$

Any $(n-1)-$ person game u in $(0,1)$ normalization can be converted to an equivalent n-person constant sum game v in $(0,1)$ normalization as follows:

$$v(S) = \begin{matrix} u(S) & \text{if } n \notin S \\ 1 - u(N - S) & \text{if } n \in S \end{matrix}$$

Here $N = \{1, 2, ..., n\}$.

Definition 12 *A game v is symmetric if $v(S)$ depends only on $|S|$.*

Definition 13 *A game v in $(0,1)$ normalization is called a simple game if*

$$v(S) \in \{0,1\} \qquad \forall S$$

Coalitions S with $v(S) = 1$ are called *winning coalitions* and those with $v(S) = 0$ are called *losing coalitions.*

Definition 14 *Let $(p_1, p_2, .., p_n)$ be a nonnegative vector and let q satisfy the relation*

$$0 < q < \sum_{i=1}^{n} p_i$$

The ***weighted majority game*** *$(q; p_1, p_2, ..., p_n)$ is defined as a simple game v in $(0,1)$ normalization where*

$$v(S) = \begin{matrix} 1 & \quad \text{if } \sum_{i \in S} p_i \geq q \\ 0 & \quad \text{else} \end{matrix}$$

Definition 15 *The set of* ***undominated imputations*** *$C(v)$ of a game v is called the* ***core*** *of a game.*

Theorem 16 *$C(v)$ is the set of n-vectors x satisfying the relations;*

$$\sum_{i \in S} x_i \geq v(S) \qquad \forall S \subseteq N$$
$$\sum_{i=1}^{n} x_i = v(N)$$

Proof. *Clearly, the first condition implies the result that*

$$x_i \geq v(\{i\}) \qquad \forall i$$

Hence any vector that satisfies both relations above is an imputation. Suppose x satisfies both relations. Let y be an n-vector satisfying the relation

$$y_i > x_i \qquad \forall i \in S$$

for some $S \subseteq N$. Then

$$\sum_{i \in S} y_i > \sum_{i \in S} x_i \geq v(S)$$

Hence there is no vector y that dominates x. Hence vectors that satisfy both relations are undominated.

Conversely, suppose we have an n-vector y that does not satisfy both relations. If

$$\sum_{i=1}^{n} y_i \neq v(N)$$

then y is not an imputation and hence not in the core. Suppose

$$\sum_{i=1}^{n} y_i = v(N)$$
$$\sum_{i \in S} y_i = v(S) - \epsilon$$

for some $\epsilon > 0$ and some nonempty set $S \subset N$. By superadditivity it follows that

$$\alpha = v(N) - v(S) - \sum_{i \in N-S} v(\{i\}) \geq 0$$

Let $|S| = s$; [note that $0 < s < n$]. Consider an n-vector z defined as follows:

$$z_i = \begin{array}{c} y_i + \frac{\epsilon}{s} \\ v(\{i\}) + \frac{\alpha}{n-s} \end{array}$$

It is easy to verify that z is an imputation and that $z \succ_S y$ and hence y can not be in the core. ∎

This result shows that the core is a closed convex polyhedral set.

Example 1 *Player 1 (seller) has a horse which is of no value to him. There are two buyers #2, #3 who want to buy the horse. #2 has a value of $90 and #3 has value of 100 for the horse. The characteristic function form for this game is*

$$v(\{i\}) = 0 \qquad \forall i$$
$$v(\{2,3\}) = 0$$
$$v(\{1,2\}) = 90$$
$$v(\{1,3\}) = v(\{1,2,3\}) = 100$$

Hence the core consists of vectors x satisfying the relations:

$$\begin{aligned} x_1 + x_2 &\geq 90 \\ x_1 + x_3 &\geq 100 \\ x_1 + x_2 + x_3 &= 100 \\ x_i &\geq 0 \qquad \forall i \end{aligned}$$

The core for this game is given by

$$C(v) = \{(t, 0, 100 - t) : 90 \leq t \leq 100\}$$

Exercise 17 *What is the non-cooperative solution to this game?*