Examples

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Example 1 *Cournot Competition:*

Consider a single commodity and there are n manufacturers. Each manufacturer decides the amount he produces. The price at which this commodity sells depends on the total quantity produced by all manufacturers and is normally assumed to a nonincreasing function of this quantity. The cost of manufacturing q units for the ith manufacturer is $c_i(q)$ and it is assumed that this is common knowledge for all manufacturers (i.e each one knowns the production cost of all others as well his own.) It is usually assumed that $c_i(q)$ is concave increasing function. How much should each produce? What would be the "equilibrium" production values? This has been called "Cournot" equilibrium but for uniformity we should call is Nash equilibrium for the Cournot application. [Cournot's work predates that of Nash.]

If the quantities produced are given by the vector $[q_1, q_2, ..., q_n]$, then the payoff $u_i(q_1, q_2, ..., q_n)$ to i^{th} manufacturer is given by

$$u_i(q_1, q_2, ..., q_n) = [q_i \cdot P(\sum_{i=1}^{n} q_i)] - c_i(q_i)$$

Now we apply the usual Nash equilibrium conditions to this payoff function. The so called "best response" function calculates the value of q_i that maximizes u_i with respect to q_i assuming all other variables are fixed. This yields the equation (from first order conditions)

$$\frac{\partial u_i}{\partial q_i} = P + q_i \frac{\partial P}{\partial q_i} - \frac{dc_i}{dq_i} = 0 \qquad \text{for } i = 1, 2, ..., n$$

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If we assume further that

$$P(\sum_{i=1}^{n} q_i) = \begin{cases} a - \sum_{i=1}^{n} q_i & \text{if } \sum_{i=1}^{n} q_i \leq a \\ 0 & \text{else} \end{cases}$$

$$c_i(q_i) = cq_i$$

In this case,

$$u_i(q_1, q_2, ..., q_n) = q_i[a - \sum_{i=1}^{n} q_i] - cq_i$$

$$\frac{\partial u_i}{\partial q_i} = a - c - [\sum_{i=1}^{n} q_i] - q_i$$

Setting $\frac{\partial u_i}{\partial q_i} = 0$, we get

$$q_i = \frac{a - c - \sum_{j \neq i} q_j}{2}$$

By symmetry, we get the solution of the above equations:

$$q_i = \frac{a - c}{n + 1} \text{ for all } i$$

Please note that this solution satisfies the relation $\sum_{i=1}^{n} q_i = (a-c)(1-\frac{1}{n+1}) \leq a$.

Example 2 *Bertrand Model of Competition*

Again there are n manufacturers; but the products they produce may not be exactly the same. However, the demand for one of these may be affected not only by its price but also the price of the products of other manufacturers and in this sense these may be substitutable products. [Think of laptops by different manufacturers.] Manufacturer i sets a unit price p_i for his product – this is his only decision variable [we assume the features of his product are predetermined]. Given this, we have n demand functions $d_i(p_1, p_2, ..., p_n)$ which determine the quantities produced. It is assumed that

$$\frac{\partial d_i}{\partial p_i} < 0$$

$$\frac{\partial d_i}{\partial p_j} > 0 \qquad \text{for } j \neq i$$

Cost of producing q units for i^{th} manufacturer is $c_i(q)$. Bertrand model assumes that the amount produced by manufacturer i is equal to the demand d_i. [One could relax this and allow the manufacter to produce any quantity no more than d_i.] The net gain to maufacturer i is given by:

$$u_i(p_1, p_2, ..., p_n) = p_i d_i(p_1, p_2, ..., p_n) - c_i(d_i(p_1, p_2, ..., p_n))$$

Now we can define Nash equilibrium in the usual manner for this model and this some times called Bertrand equilibrium. If we have linear demand functions:

$$d_i(p_1, p_2, ..., p_n) = a_i - b_i p_i + \sum_{j \neq i; j=1}^{n} r_{i,j} p_j; r_{i,j} > 0; j \neq i$$

Suppose that the cost is also linear (with no fixed charges) – i.e.:

$$c_i(q) = c_i q$$

In this case we can write the gain vector as:

$$u_i(p_1, p_2, ..., p_n) = a_i p_i - b_i p_i^2 + p_i \sum_{j \neq i; j=1}^{n} r_{i,j} p_j - c_i a_i + c_i b_i p_i - c_i \sum_{j \neq i; j=1}^{n} r_{i,j} p_j$$

Setting $\frac{\partial u}{\partial p_i} = 0$ (to get the best response functions) we get:

$$a_i - 2b_i p_i + c_i b_i + \sum_{j \neq i; j=1}^{n} r_{i,j} p_j = 0$$

which in turn reduces to

$$2b_i p_i = a_i + c_i b_i + \sum_{j \neq i; j=1}^{n} r_{i,j} p_j$$

We can write this in matrix form as follows: Let T be an $n \times n$ matrix whose entries are:

$$T_{i,j} = \begin{cases} 2b_i & \text{if } j = i \\ -r_{i,j} & \text{else} \end{cases}$$

Let α be an n-vector with

$$\alpha_i = a_i + b_i c_i$$

Then these equations become:

$$\begin{aligned} Tp &= \alpha \\ p &= T^{-1}\alpha \end{aligned}$$

assuming this inverse exists. A matrix with off diagonals negative, diagonals positive and whose inverse exists is called an M matrix and it is known that this inverse is a positive matrix (all elements are postive). The case when $n = 2$ and $a_i = a; b_i = 1; c_i = c < a; 0 < r_{i,j} = b < 2$ is discussed in the book by Gibbons. Here we get:

$$\begin{aligned} p_i &= \frac{1}{2}(a + c + bp_j) \\ p_j &= \frac{1}{2}(a + c + bp_i) \end{aligned}$$

and hence

$$p_i = p_j = \frac{a + c}{2 - b}$$

Example 3 *Final-Offer-Arbitration: (RG):*

Two major forms of arbitration: conventional and final-offer arbitration. In the final-offer form, two parties (the firm and the union) make wage offers. There is an arbitrator who selects one of the two. In conventional arbitration, the arbitrator is free to select any settlement that he chooses (possibly different from the one on offer by the two parties).

Suppose the firm's offer is w_f and the union's offer is w_u (and we assume that $w_f < w_u$). Assume that the arbitrator has an indeal settlement in mind (known only to him) and this value is x. In the final-offer arbitration, the arbitrator selects the offer that is closer to the ideal of the two and if they are equidistant, it does not amtter which one is selected. The two parties know the value of x only in the form of a probability distribution $F(x)$ (whihc is assume to be continuous and differentiable). The density function of this distribution is $f(x)$. It is easy to see that the expected value of the settlement wage as a function of the offers w_f by the firm and w_u (with $w_f < w_u$) is given by:

$$w_f F(\frac{w_f + w_u}{2}) + w_u[1 - F(\frac{w_f + w_u}{2})]$$

The firm wants to minimize this quantity and the union wants to maximize it. If (w_f^*, w_u^*) forms a Nash equilibrium for this game it must satisfy the relations:

$$(w_u^* - w_f^*) \cdot \frac{1}{2} f(\frac{w_f^* + w_u^*}{2}) = F(\frac{w_f^* + w_u^*}{2})$$
$$(w_u^* - w_f^*) \cdot \frac{1}{2} f(\frac{w_f^* + w_u^*}{2}) = 1 - F(\frac{w_f^* + w_u^*}{2})$$

These are the best response relations for the two parties. This implies that

$$F(\frac{w_f^* + w_u^*}{2}) = \frac{1}{2}$$

and hence

$$w_u^* - w_f^* = \frac{1}{f(\frac{w_f^* + w_u^*}{2})}$$

The gap between two offers must equal the reciprocal of the density at the median. If the distribution is $N(\mu, \sigma)$, then this works out to

$$w_u = \mu + \sigma\sqrt{\frac{\pi}{2}}$$
$$w_f = \mu - \sigma\sqrt{\frac{\pi}{2}}$$

Example 4 *The Problem of the Commons: (RG)*

This concerns the overutilization of public resources. Consider n farmers in a village that graze their goats on the village green. Let the number of goats that i^{th} farmer own be g_i. Total number of goats that graze on the green is

$$G = \sum_{i=1}^{n} g_i$$

The cost per goat to the farmer of owning is c. The value per goat of grazing in the green is dependent on the number of goats on the green and is given by $v(G)$. It is assume that this is decreasing and concave function. During the spring the farmers simultaneously choose the numebr of goats to own. The payoff to farmer i is given by

$$u_i(g_1, g_2, ..., g_n) = g_i v(G) - c g_i$$

Now we look for a Nash equilibrium vector $(g_1^*, g_2^*, ..., g_n^*)$. The first order conditions are

$$v(G^*) + g_i^* v'(G^*) - c = 0$$

Adding these and dividing by n we get

$$v(G^*) - \frac{1}{n} G^* v'(G^*) - c = 0$$

In contrast, the social optimum (maximize the village's total payoff) is given by $\max_G [Gv(G) - cG]$ which is obtained by solving the equation:

$$v(G^{**}) - G^{**} v'(G^{**}) - c = 0$$

Using the properties of the function $v(G)$ we get $G^* > G^{**}$. This says that too many goats are grazed in the Nash equilibrium compared to the social optimum – higher use of public resources.