1. Assume that \( N = \{1, 2, \ldots, n\} \) is the set of players in a game with \( n > 1 \). Let the set of pure strategies for player \( i \) be denoted by \( S_i; i = 1, 2, \ldots, n \). Let the payoff function for player \( i \) be defined by \( \pi_i : S_1 \times S_2 \times \ldots \times S_n \rightarrow \mathbb{R} \). A strategy \( s_i \in S_i \) is strictly dominated by \( s'_i \in S_i \) if
\[
\pi_i(s_i, s_{-i}) < \pi_i(s'_i, s_{-i}) \quad \forall s_{-i} \in \prod_{j \neq i} S_j
\]
Elimination of a strictly dominated strategy \( s_i \) changes to the set \( S_i \) to \( S'_i = S_i - \{s_i\} \). Iterated elimination for each player until no further elimination is possible results in players sets \( S'_i; i = 1, 2, \ldots, n \). With respect to this process answer the following questions:

(a) Can any of these sets \( S'_i \) become empty if none of the sets \( S_i \) are empty?

**Solution:** A strategy is eliminated by another which remains in the set after elimination. Hence these sets cannot become empty.

(b) Is the final outcome the same regardless of the order in which these are executed?

**Solution:** Strict domination continues to hold after the elimination since the requirements are less for domination after elimination. Hence the order does not matter.

(c) Would any Nash equilibrium of the original game be eliminated in this process?

**Solution:** Strictly dominated strategies cannot be part of an equilibrium profile by the definition of equilibrium and hence this is not possible.

2. Show that strictly dominant equilibrium is unique for an \( n \)-player game when it exists.

**Solution:**
Recall that a strategy \( s_i \) is a strictly dominant strategy for player \( i \) if it satisfies the relation:
\[
\pi_i(s_i, s_{-i}) > \pi_i(s'_i, s_{-i}) \quad \forall s_{-i}, \text{ and } \forall s'_i \neq s_i
\]
Clearly, there can be at most one such \( s_i \) for each player. For if \( s_i \) and \( s'_i \) are both dominant, then we have
\[
\pi_i(s_i, s_{-i}) > \pi_i(s'_i, s_{-i}) \quad \forall s_{-i},
\]
\[
\pi_i(s'_i, s_{-i}) > \pi_i(s_i, s_{-i}) \quad \forall s_{-i},
\]
which is not possible. A dominant equilibrium consists of a combination of strictly dominant strategies for each player. Hence the result.
3. Show that a strictly dominant equilibrium is also a Nash equilibrium.

Solution:
Let \((s_1, s_2, ..., s_n)\) be a strictly dominant equilibrium. Then

\[ \pi_i(s_i, s_{-i}) > \pi_i(s_i', s_{-i}) \quad \forall s_{-i}, \text{and} \forall s_i' \neq s_i \]

Hence player \(i\) does not have any incentive to use any other strategy if all other players use their part of the dominant strategy profile. Hence, for each player \(i\), \(s_i\) is a best response to \(s_{-i}\). Hence \((s_1, s_2, ..., s_n)\) is a Nash equilibrium.

4. Show that when a strictly dominant equilibrium exists, then the Nash equilibrium is unique.

Solution:
Let \((s_1, s_2, ..., s_n)\) be a strictly dominant equilibrium. Let \((s_1', s_2', ..., s_n')\)[\(\neq (s_1, s_2, ..., s_n)\)] be a Nash equilibrium. Suppose \(s_i \neq s_i'\). Then

\[ \pi_i(s_i, s_{-i}^*) > \pi_i(s_i', s_{-i}^*) \quad \forall s_{-i}^*, \text{and} s_i' \neq s_i \]

Hence

\[ \pi_i(s_i, s_{-i}') > \pi_i(s_i', s_{-i}') \quad s_i' \neq s_i \]

This contradicts the assumption that \((s_1', s_2', ..., s_n')\) is a Nash equilibrium. Hence the result.

5. Is it possible to have a strictly dominant equilibrium that consists of mixed strategies? If yes provide an example; if not show why.

Solution:
Let \(p_i\) be a strictly dominant mixed strategy (that is NOT a pure strategy) for player \(i\) which is given by a nonnegative vector \(x\) with as many components as there are pure strategies for player \(i\) and with \(\sum_{j=1}^{k} x_j = 1\). So this mixed strategy "uses" the pure strategy \(s_i^j\) with probability \(x_j; j = 1, 2, ..., k\). Please note that, since this a mixed strategy, \(0 < x_j < 1\) for all \(j\). Since \(p_i\) is a strictly dominant strategy for player \(i\),

\[ \pi_i(p_i, s_{-i}) > \pi_i(s_i^j, s_{-i}) \quad \forall s_i^j; s_{-i} \]

Multiplying the \(j^{th}\) of these with \(x_j\) and adding we get

\[ \pi_i(p_i, s_{-i}) > \pi_i(p_i, s_{-i}) \]

which is a contradiction. Hence, there can not be mixed strategy for any player that strictly dominates all pure strategies for this player. Hence there can not be mixed strategy profile that is strictly dominant equilibrium.
6. What happens to the set of Nash equilibria when dominated strategies are eliminated?

**Solution:**

Any strictly dominated pure strategy for any player can not be a part of a Nash equilibrium profile. Suppose \( s^d_i \) is a strictly dominated pure strategy for player \( i \). This means that there is a strategy \( s_i \) for player \( i \) such that

\[
\pi_i(s^d_i, s_{-i}) < \pi_i(s_i, s_{-i}) \quad \forall s_{-i}
\]

This makes it clear that \( s^d_i \) can not be a part of a Nash equilibrium profile.

But if a strategy is weakly dominated, then it might be a part of a Nash equilibrium profile. See:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

So elimination of such a strategy might remove a Nash equilibrium profile. But this can not alter the existence of a Nash equilibrium (possibly mixed) in the sense at least one of the Nash equilibrium of the original problem survives as a Nash equilibrium of the reduced problem.

7. If there is a strategy profile \((s_1, s_2, ..., s_n)\) for an \(n\)-player game such that the payoff for each player for this profile is strictly larger than the corresponding payoff for any other profile, is this profile a dominant equilibrium? Is it a Nash equilibrium?

**Solution:**

Consider the following example:

\[
\begin{array}{ccc|ccc|ccc}
1 & 2 & \rightarrow & L & C & R \\
T & [0, 4] & [4, 0] & [5, 3] \\
M & [4, 0] & [0, 4] & [5, 3] \\
\end{array}
\]

This game has no dominant strategies for either player and hence no dominant equilibrium. However, \([B, R]\) is a Nash equilibrium. (This is easy to show and is left to the reader!). Proof of the general result: Since we are given that \( \pi_i(s^*_1, s^*_2, ..., s^*_{i-1}, s_i^*, s^*_{i+1}, ..., s^*_n) > \pi_i(s_1, ..., s_n) \) for \((s_1, ..., s_n)\), it should be clear that for each player \( s^*_i \) is the best response against \( s^*_{-i} \). Hence this is a Nash equilibrium.