Assignment #2:
Please check my solutions for correctness and any missing points!

1. Hotelling’s Model (RG): Consider a population uniformly distributed along an ideological spectrum \([0, 1]\). Each of two candidates chooses (simultaneously) a platform represented by an ideology value \(x \in [0, 1]\). The voters observe the platforms and choose to vote for the candidate whose platform is closest to their ideological beliefs. Suppose each candidate cares only about winning the election. What is (if any ) pure strategy Nash equilibrium for this game? [Optional] What happens if there are three (or more) candidates?

Solution:
Suppose \(n = 2\) first. There are two ways of modeling this problem: In one, each player wants to maximize the number of votes for himself; in the other, each player wants to maximize the probability of winning. [Q: Does a player who wins plurality win the election?] We take the first approach here. The payoff to player 1 if the profile is \((x, y)\) with \(x, y \in [0, 1]\) is given by
\[
\pi_1(x, y) = \begin{cases} 
\frac{x+y}{2} & \text{if } x \leq y \\
1 - \frac{x+y}{2} & \text{if } x > y
\end{cases}
\]
\[
\pi_2(x, y) = 1 - \pi_1(x, y)
\]
Suppose player 2 selects \(y \neq \frac{1}{2}\). There is no best response for player 1! Hence there is no pure strategy profile that is a Nash equilibrium of the type \((x, y)\) where \(y \neq \frac{1}{2}\). Similar arguments are true for the other player. It is easy to verify that \((\frac{1}{2}, \frac{1}{2})\) is a pure strategy profile that is a Nash equilibrium. Hence the only Nash equilibrium among pure strategy profiles is \((\frac{1}{2}, \frac{1}{2})\). Please note that this is not a dominant equilibrium and hence no dominant equilibrium exists for this game.

2. There are two firms with one job opening each. Firm \(i\) offers a wage of \(w_i\) [Here \(i = 1, 2\)]. There are two workers each of whom may apply to either firm and they choose this simultaneously. If exactly one worker applies to a firm, he gets the job. If both apply to the same firm, then each has a probability of 0.5 to get the job. The other will, in this case, be without a job. Solve for Nash equilibrium. [You may assume that \(\frac{1}{2}w_1 < w_2 < 2w_1\)].

Solution:
The payoff matrix looks like:

<table>
<thead>
<tr>
<th></th>
<th>Apply to Firm #1</th>
<th>Apply to Firm #2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apply to Firm #1</td>
<td>(\frac{1}{2}w_1, \frac{1}{2}w_1)</td>
<td>((w_1, w_2))</td>
</tr>
<tr>
<td>Apply to Firm #2</td>
<td>(w_2, w_1)</td>
<td>(\frac{1}{2}w_2, \frac{1}{2}w_2)</td>
</tr>
</tbody>
</table>

There are no dominant equilibria for this game. There are two pure Nash equilibria: \((#1, #2)\) and \((#2, #1)\). There is also a mixed strategy equilibrium. Here player I uses the strategy \((\frac{2w_2-w_1}{w_1+w_2}, \frac{2w_1-w_2}{w_1+w_2})\) and player II uses the strategy \((\frac{2w_1-w_2}{w_1+w_2}, \frac{2w_2-w_1}{w_1+w_2})\).
3. [Problem of Commons][RG]: There are $n$ farmers in a village who graze their goats in the village commons. Farmer $i$ selects (independent of others), to have $g_i$ goats; $i = 1, 2, ..., n$. Cost of raising a goat for farmer $i$ is $c_i$ per goat independent of the number of goats he raises. The value that he gets depends on what others are doing as well and is described by a function $v(G)$ where $G = \sum_{i=1}^{n} g_i$ and $v(x)$ is decreasing and concave $\left[\frac{dv(x)}{dx} < 0; \frac{d^2v(x)}{dx^2} < 0\right]$. Moreover, $v(x) > 0$ for $x < G_{\text{max}}$ and $v(x) = 0$ for $x \geq G_{\text{max}}$. Assume that farmers can select fractional number of goats to raise. Show how to find a Nash equilibrium if $c_i = c$ for all $i$.

Solution: The value in this problem is per goat raised. For the strategy profile $(g_1, g_2, ..., g_n)$ payoff to player $i$ is $\pi_i(g) = g_i \left[\nu(G) - c\right]$. Thus if profile $(g^*_1, g^*_2, ..., g^*_n)$ is a Nash equilibrium, we must have

$$\frac{\partial \pi_i(g)}{\partial g_i} = 0 \text{ for } g = g^*$$

which yields the equation:

$$\nu(G^*) - c + g^*_i \nu'(G^*) = 0$$

where $G^*$ satisfies:

$$\nu(G^*) - c + \frac{1}{n} \nu'(G^*) = 0$$

and $g^*_i = \frac{1}{n} G^*$. This is a symmetric game and has a symmetric equilibrium.

4. First consider a two player game in which the number of pure strategies for player I is $m$ and that for player II is $n$. Let the payoff matrices be $A$ and $B$ [each of which is $m \times n$]. Show the following statements are true:

(a) If $(x^*, y^*)$ is a Nash equilibrium pair of mixed strategies; $[(x^*)_t A y^*_t, (x^*)_t B y^*_t] = (u^*, v^*)$, then

$$x^*_i > 0 \Rightarrow \sum_{j=1}^{n} a_{i,j} y^*_j = u^*$$

$$y^*_j > 0 \Rightarrow \sum_{i=1}^{m} b_{i,j} x^*_i = v^*$$

Solution: Since $(x^*, y^*)$ is a Nash equilibrium, we have

$$u^* = (x^*)_t A y^* \geq x^t A y^* \quad \forall x$$

Hence it follows by using pure strategies for $x$ we have

$$u^* \geq \sum_{j=1}^{n} a_{i,j} y^*_j \quad \forall i$$
By multiplying the $i^{th}$ of these by $x_i^*$ and adding we get:

$$u^* = u^* \sum_{i=1}^{m} x_i^* \geq (x^*)^t Ay^* = u^*$$

Hence equality holds throughout this equation/inequality. Strict inequality holds if there is some index $i$ such that both $x_i^* > 0$ and $u^* > \sum_{j=1}^{n} a_{i,j} y_j^*$. Hence the result. The second part is similar.

(b) Suppose $(x^0, y^0)$ is a strategy profile such that

$$x_i^0 > 0 \Rightarrow \sum_{j=1}^{n} a_{i,j} y_j^0 = u^0 = \max_{k} (\sum_{j=1}^{n} a_{k,j} y_j^0)$$

$$y_j^0 > 0 \Rightarrow \sum_{i=1}^{m} b_{i,j} x_i^0 = v^0 = \max_{l} (\sum_{i=1}^{m} b_{i,l} x_i^0)$$

then, $(x^0, y^0)$ is an equilibrium pair of strategies.

**Solution:** For such a pair $(x^0, y^0)$ we have:

$$u^0 = (x^0)^t Ay^0 \geq x^t Ay^0 \quad \forall x$$

$$v^0 = (x^0)^t By^0 \geq (x^0)^t Ay \quad \forall y$$

Hence the result follows.

[This is often stated as "only strategies that have positive probability in an equilibrium pair are those that are "best " responses against the opponent’s strategy. This actually generalizes to $n$–player games with appropriate modifications].

5. A Nash equilibrium is called a strict Nash equilibrium if any player who deviates strictly loses by doing so if all others remain at their part of the equilibrium.

(a) Prove that if the process of iterated elimination of strictly dominated strategies yields a unique profile, then this is a strict Nash equilibrium.

**Solution:** If at any step strategy $x \in A_i$ for player $i$ strictly dominates strategy $y \in A_i$, then

$$\pi_i(x, x_{-i}) > \pi_i(y, x_{-i}) \quad \forall x_{-i} \in A_{-i}$$

where $A_{-i}$ is the product set of the remaining strategies for all other players. Each strategy in $A_{-i}$ for each player $j \neq i$, strictly dominates the remaining strategies for that player at a previous step. Hence similar inequality holds for them also. Hence this profile that remains if unique, is a strict nash equilibrium.
(b) If there is a strict Nash equilibrium, then none of the strategies in the profile corresponding to this equilibrium can be eliminated by iterated domination under either strict or weak dominance.

Solution: reasoning for this is similar to the above.

6. First Price Auction: There is an object being auctioned; there are \( n \) bidders; the value of the object to bidder \( i \) is known only to bidder \( i \). Each bidder submits a sealed bid to the auctioneer. The object is sold to the highest bidder and (s)he pays the amount bid to the auctioneer. If there are "ties" these are broken by a fair lottery – equal probabilities to each highest bidder. For bidder \( i \), is there a weakly dominating strategy? Does the strategy-profile under which each bidder bids his/her true (private) value for the object weakly dominate all other profiles? What is a Nash equilibrium for this problem?

Solution: This problem has been withdrawn. It requires a common probability distribution on the values from which each player’s value is drawn.