Assignment #3

1. Analyze the following game: State whether this game is one of (i) perfect information; (ii) perfect recall; (iii) Subgame perfect equilibria for this game.

Solution:
Since there is an information set for player II that has more than one node, this game is not one of perfect information. There is no information in the problem that contradicts the statement that each player knows all his past actions at each of his nodes, and all that he knew before. Hence this is a game of perfect recall. Subgames other than the whole game are: (a) The subgame corresponding to the subtree rooted at the node where it is player II’s turn to act and his choices are U and V, and (b) The subgame corresponding to the subtree rooted at node where it is a simultaneous move by both players. These nodes are singleton information sets and no information set of the whole game is split by these subtrees.

Pure Strategies:
For player I: \{L, A|V\}, \{L, B|V\}, \{R, A|V\}, \{R, B|V\}. The second part of each corresponds to substrategy for subgames described in (a) and (b).
For player II: \{U|R, C\}, \{U|R, D\}, \{V|R, C\}, \{V|R, D\}. These are substrategies for subgame (a) and substrategies for subgame (b) are: \{C\}, \{D\}.

Subgame perfect equilibrium for subgame (b) [the lowest level]:
This is a zero sum two person game that is symmetric with two strategies for each player. There are no pure equilibria and the only mixed equilibrium has probability vectors for each player equal to \(\left(\frac{1}{3}, \frac{1}{3}\right)\). The unique payoff for this equilibrium is \([0,0]\). By working backwards this leads to
the game:

```
I
L R
[2,2] U V
[3,1]
[0,0]
```

For the only subgame of this game that corresponds to the rooted tree at the node where it is player II’s turn to act, U dominates V. Hence backward induction leads to the game:

```
L I
R [2,2] [3,1]
```

In this game only player I acts (once) and R dominates L. Hence the subgame perfect equilibrium for the entire game consists of the following strategies for the two players:

For player I: \{R, \(\frac{1}{2}, \frac{1}{2}\)\} | V}; for player II: \{U, \(\frac{1}{2}, \frac{1}{2}\)\} | V \} The equilibrium outcome payoff vector is \[3, 1\] that corresponds to the result of \{R, U\}.

2. Analyze the following games: (known as Centipede)

```
I
II
I
II
I
[5,5]

[1,0]  [0,1]  [3,0]  [2,4]  [6,3]

1  2  3  4  5  2

```

Solution:

We indicate actions that take us to the right by R and those that take us down by D for each player.

First Game: In his last move (and the subgame that corresponds to it), for player I (whose turn it is): D dominates R and hence this subgame can
be replaced by the payoff vector \([6,3]\) and we get a "reduced" centipede which looks like:

\[
\begin{array}{c}
\text{I} \\
[1,0] \\
\text{II} \\
[0,1] \\
\text{I} \\
[3,0] \\
\text{II} \\
[2,4] \\
\end{array}
\]

Now for the last choice of player II, D dominates R and this subgame of the reduced game can be replaced by the payoff vector \([2,4]\) to get a second reduced game that looks like:

\[
\begin{array}{c}
\text{I} \\
[1,0] \\
\text{II} \\
[0,1] \\
\text{I} \\
[3,0] \\
\text{II} \\
[2,4] \\
\end{array}
\]

Again D dominates R for player I in his last subgame. In each of their turns for both players, D dominates R and hence the equilibrium outcome payoff is \([1,0]\) and subgame perfect equilibrium (unique) consists of the following strategies:

Player I: \{D, D|R|R, D|R|R|R|R\} and that of player II: \{D|R, D|R|R|R\).

**Second Game:** By a similar analysis we get the only subgame perfect equilibrium to be for each player (if and when his turn comes) make their only choice to be R. The final outcome results in the payoff vector which pays each player 2. The analysis is simple; but this is used as an example to try to show that this concept is problematic in the sense that each player must know that all subsequent players are rational (and each knows his successors are rational etc.) and there is no chance that they may do the other action. This faith may be a decreasing function of the length of this centipede.

3. In the following game, what are the subgames? Find a Subgame perfect
Nash equilibrium.

Note that this game has no "proper" subgame; no subgame can split an information set. Player III is indifferent to his choices regardless of what the other players do. Thus, we can allow any strategy \((z_L, z_R = 1 - z_L)\) with \(z_L \in [0, 1]\) for player III in any equilibrium profile from player III's perspective. From the perspective of players I and II, this game is equivalent [under the condition that III uses \((z_L, z_R = 1 - z_L)\)] to the game:

This "new" game has a subgame at the node when it is player II's turn to choose. For player II, at this point, D dominates A if and only if \(z_R > \frac{1}{3}\). So we get two possibilities shown below (the first if \(z_R < \frac{1}{3}\) and the second
if \( z_R > \frac{1}{3} \):

Since \( [z_R < \frac{1}{3}] \Rightarrow [z_L > \frac{1}{3}] \), in the first case for player I D dominates A. In the second case, player I is indifferent (under Nash) between D and A. If \( z_R > \frac{1}{3}; z_L > \frac{1}{3} \), player I is indifferent between A and D under the assumption that player II (being rational) will select D. But if player I is concerned about "trembles" of player II, he would prefer to use D. So one set of equilibria that are trembling hand perfect are: \([\{\text{Player I: D}\}, \{\text{Player II: D}\}, \{\text{Player III : (}z_L, z_R\text{)} \text{ with } z_L > \frac{1}{3}; z_R > \frac{1}{3}\}]\). Another set is given by: \([\{\text{Player I: D}\}, \{\text{Player II: A}\}, \{\text{Player III: (}z_L, z_R\text{)} \text{ with } z_L > \frac{1}{3}; z_R < \frac{1}{3}\}]\). One more is given by \([\{\text{Player I: A}\}, \{\text{Player II: D}\}, \{\text{Player III: (}z_L, z_R\text{)} \text{ with } z_L < \frac{1}{3}; z_R > \frac{1}{3}\}]\). If player III selects \((\frac{2}{3}, \frac{1}{3})\), Player I is indifferent (so use any vector \((x_D, x_A)\) which has nonnegative components that add to 1); but player II selects D\|A. If player III selects \((\frac{1}{3}, \frac{2}{3})\), Player I selects D and player II is indifferent (and may choose any vector \((y_D, y_A)\) which has nonnegative components that add to 1). All of these are "trembling hand" perfect equilibria. I think (!) this is all equilibria for this game.

4. In the following game (O. Board) refers to the Trojan war. G stands for Greeks, T for Trojans. s stands for stay put near Troy; h stands for go home; t stands for sail behind a nearby island of Tenedos. The red arrow (directed) stands for the notion that Trojans believe that Greeks have gone home when indeed they are near Troy behind Tenedos. When Greeks take actions s or h, Trojans know this to be the case. The only confusion is when action t is taken. This extends the notion of extensive
form games and alters the notion of information sets. Analyze this game:

There is a similar game concerning investment banks. With these examples, the relation that binds nodes into the same information set is no longer bidirectional.

**Solution:**

One subgame corresponds to the subtree starting at the node of T when G selects s. Here c dominates o and the resulting payoff vector is [-1,0]. There is another subgame corresponding to the subtree rooted at the node where T moves resulting from the choice of h by G. Here o dominates c and the resulting payoff vector is [0,1]. If G selects t, T is under the "false" impression that G has selected h and hence T uses his dominant strategy o. G knows all this and so selects his "dominant" strategy t. So the only subgame perfect equilibrium for this game is the following: [G selects t; T selects {c|s, o|h}] resulting in the payoff vector [1,-1].

5. The following is intended to make you read the work of G. Bonanno on the role of memory of past actions and memory of past knowledge in perfect recall.

**Memory of past knowledge:** If at some point \( t \) in time, \( \phi \) is known to be true to player i, then at point in time \( t' > t \), if player i knows that the statement \( \phi \) was known to him to be true at time \( t \).

**Memory of past actions:** [Action Recall]: At any node in the game tree where it is player i’s turn to move, player i knows the set of all his previous actions (but not necessarily the order in which he took them). Hence, for two nodes to be in the same information set of a player i, these sets of his past actions must be the same. He is allowed to "forget" the order – i.e. his memory of past knowledge may not be complete.

Perfect recall needs both. these concepts are illustrated in the following situations:
(a) Consider the following maze:

There are two points of entry: A and B. Player 1 is given the map and taken by player 0 to one of the entry points. "Player 1 is told which entry point it is that he is taken to". He takes a right and then a left. But by now he forgets where he started and so is confused as to whether he is at x or y. Draw the corresponding "game" tree (with all turns denoted by actions of player 1). Here he recalls all his actions (even the order!) but forgets previous knowledge.

**Solution:**

A stated above, Player 1 knows where he is taken to and in this case the game tree looks like:

Nodes with same color are in the same information set. But now suppose, player 1 does not know where he enters (he is blindfolded
and let in), then the game tree looks like:

Nodes of the same color are in the same information set. The first of these is not a game of perfect recall since nodes in the same information set do not have predecessors in the same information set. The tree in the second diagram is one of perfect recall! The player not only did not know if he was at X or Y even though he recalls perfectly his actions, he never knew where he entered – so there is nothing that he does not know now that he did earlier.

(b) Is the following game one of perfect recall? Is it true that there a behavioral strategy that is equivalent to a mixed strategy for all mixed strategies for this game?

Before analyzing this game, let us consider its pure strategies for its only player player I. They consist of one choice between {a,b} at I1 followed by one choice between {c,b} at I2, a similar choice at
I3 but independent of choice at I2, one choice between \{d,e\} at the information set containing \{I4, I5\}. Thus there are 16 pure strategies. Let us denote these by \{a,c,a,d\}, \{a,c,a,e\}, \{a,c,c,d\}, \{a,c,c,e\}, \{a,b,a,d\}, \{a,b,a,e\}, \{a,b,c,d\}, \{a,b,c,e\}, \{b,c,a,d\}, \{b,c,a,e\}, \{b,c,c,d\}, \{b,c,c,e\}, \{b,b,a,d\}, \{b,b,a,e\}, \{b,b,c,d\}, \{b,b,c,e\}. Behavioral strategies correspond to a mixture at I1 followed by an independent mixture at I2, an independent mixture at I3 and finally an independent mixture at the information set \{I4, I5\}. Suppose these mixtures are specified by

\[
p_a + p_b = 1 \quad p_a \geq 0; p_b \geq 0
\]

for I1.

\[
q_c + q_b = 1 \quad q_c \geq 0; q_b \geq 0
\]

for I2.

\[
r_a + r_b = 1 \quad r_a \geq 0; r_b \geq 0
\]

at I3. And finally,

\[
s_d + s_e = 1 \quad s_d \geq 0; s_e \geq 0
\]

at the information set \{I4, I5\}. Overall mixed strategy corresponding to this behavior strategy has \(x_{acad} = p_a q_c r_a s_d\) and so on. The resulting distribution at the terminal nodes in both is \(\{p_aq_c, p_br_c, p_aqs_d, p_aqs_e, p_brs_d, p_brs_e\}\) respectively on \[T1, T2, T3, T4, T5, T6\]. Thus, corresponding to a behavior strategy there is a mixed strategy with resulting probability distribution on terminal nodes the same.

NO:

(c) Suppose there is a mixed strategy \(x_{acad}, x_{acae}, ..., x_{bbce}\) on the pure strategies \{a,c,a,d\}, \{a,c,a,e\}, \{a,c,c,d\}, \{a,c,c,e\}, \{a,b,a,d\}, \{a,b,a,e\}, \{a,b,c,d\}, \{a,b,c,e\}, \{b,c,a,d\}, \{b,c,a,e\}, \{b,c,c,d\}, \{b,c,c,e\}, \{b,b,a,d\}, \{b,b,a,e\}, \{b,b,c,d\}, \{b,b,c,e\}. Under this mixed strategy, the probability of
passing via various nodes in the tree are:

\[
\begin{align*}
I1 & : 1 \\
I2 & : x_a \\
I3 & : x_b \\
\{I4, I5\} & : x_{ab} + x_{be} \\
T1 & : x_{ac} \\
T2 & : x_{bc} \\
T3 & : x_{bed} \\
T4 & : x_{bec} \\
T5 & : x_{bed} \\
T6 & : x_{bec}
\end{align*}
\]

Now we construct the corresponding behavior strategy as follows:

\[
\begin{align*}
p_a &= x_a \\
p_b &= x_b \\
q_c &= \frac{x_{ac}}{x_a} \\
q_b &= \frac{x_{bc}}{x_b} \\
r_a &= \frac{x_{bc}}{x_a} \\
r_c &= \frac{x_{be}}{x_b} \\
s_d &= \frac{x_{bed} + x_{bed}}{x_{ab} + x_{be}} \\
s_e &= \frac{x_{bec} + x_{bec}}{x_{ab} + x_{be}}
\end{align*}
\]

assuming none of these denominators is zero. If not, those quantities corresponding zero denominators can be chosen arbitrarily subject to nonnegativity and sum equal to 1. You can check that this behavior strategy corresponds to the mixed strategy we started with. In this sense, this game acts like a game with perfect recall. So instead of requiring that the order in which actions take place be the same for each node in the same information set, we only require that the sets of actions be the same. In the old way of defining perfect recall this would not have been a game of perfect recall, but in the "modern" definition it is.

6. This problem is inspired by the work of Ms. Liying Mu and concerns milk production in China: In the general version there are \( n \) farmers who are the first level suppliers of milk which is then collected by a collection center (called the milk sheds). These collection centers then mix this
milk from the various farmers and send to the main center (called the firm) which distributes and markets to the customers. The firm pays the collection centers an amount that depends on the quantity and quality of milk the sent to the firm by the collection center. Just as there are $n$ farmers subscribing to a collection center, there are several collections centers subscribing to a firm.

In the simple version there are two players: a farmer and a collection center. The farmer can either produce milk of high quality or milk of lower quality. Since milk of high quality fetches a unit price of $w_H > w_L$ (the price fetched by a unit of low quality milk), there is temptation for the farmer to assert that his milk is of high quality regardless of whether this is true or not. To find out if the farmer is telling the truth, the collection center has the option of testing the milk at a cost of $t$ per test. The test is supposed to be accurate. The collection center has an option of charging a penalty if it finds that the farmer has misrepresented his product. The cost to the farmer of producing low quality milk is $c_L < c_H$ (the price of producing high quality milk). There are at least two ways of depicting this game. Please note that the collection agency is aware of what the farmer tells them but not the true condition of the milk unless it tests. The order of the process is the following from the perspective of the collection center: Farmer brings milk and claims it is of certain quality. Then the firm decides whether to test or not. From the perspective of the farmer it could be either to decide first what to tell the collection center and then decide what to produce or the reverse. The collections center gets paid by the firm an amount $p_H$ if the quality is high and an amount $p_L < p_H$ if the quality is low. Show possible extensive forms; show subgames; pure strategies for the two players; and find Nash equilibria and if possible subgame perfect Nash equilibria. Does this example make you refine the notion of subgames?

**Solution:**

The following diagram shows two possible extensive forms for this game.
depending on which action the farmer takes first.

In the first form, the only subgame is the whole game since subtrees corresponding to subgames are not allowed to split information sets (nodes with the same color except white). In the second diagram, we have three subgames – two of which are proper subgames. In each of these proper subgames, we get a $2 \times 2$ game.

In the subgame corresponding to $L_c$ in which the farmer claims that he has given low quality milk, $L_P$ strictly dominates $H_P$ since $w_L - c_L > p_L - c_P$. Given this, the for the collection center, $NT$ dominates $T$ since $p_L - w_L - t < p_L - w_L$. Hence for this subgame, the only Nash equilibrium is $[L_P, NT]$ giving a payoff vector $[w_L - c_L, p_L - w_L]$ – the first of these is to farmer and the second to collection center. The payoff matrix for this game is shown below:

<table>
<thead>
<tr>
<th></th>
<th>$T$</th>
<th>$NT$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_P$</td>
<td>$[w_L - c_L, p_L - w_L - t]$</td>
<td>$[w_L - c_L, p_L - w_L]$</td>
</tr>
<tr>
<td>$H_P$</td>
<td>$[w_L - c_P, p_P - w_L - t]$</td>
<td>$[w_L - c_P, p_P - w_L]$</td>
</tr>
</tbody>
</table>

In the other subgame where the farmer claims $H_C$ the payoff matrix looks like:

<table>
<thead>
<tr>
<th></th>
<th>$T$</th>
<th>$NT$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_P$</td>
<td>$[w_L - c_L - \delta, p_L - w_L - t + \delta]$</td>
<td>$[w_P - c_L, p_L - w_P]$</td>
</tr>
<tr>
<td>$H_P$</td>
<td>$[w_P - c_P, p_P - w_P - t]$</td>
<td>$[w_P - c_P, p_P - w_P]$</td>
</tr>
</tbody>
</table>

Here we may need to solve the two person game since there may be no domination. The most likely scenario is the case with $w_P - w_L > t - \delta$ since most likely $\delta > t$ and also $w_P - w_L > c_P - c_L - \delta$ since most likely $w_P - c_P > w_L - c_L$. In this case we get only mixed equilibrium. I leave the rest to you to solve this and find the subgame perfect solution under various assumptions on the values of the parameters.