Solution #1:

1. Show that $\Omega(g(n)) \cap o(g(n))$ is the empty set.

   Solution: Suppose $f(n) = o(g(n))$. This implies that $\lim_{n\to\infty}[\frac{f(n)}{g(n)}] = 0$. If $f(n) = \Omega(g(n))$, then the previous limit can not be equal to 0. Hence the result follows.

2. Does the statement $[f(n) = O(g(n))]$ imply the statement $[2^{f(n)} = O(2^{g(n)})]$? Is the converse true? Give proofs or counter examples.

   Solution: Let $f(n) = \lg[n^n] = n \lg n$;and $g(n) = \lg(n!)$. We have shown that $f(n) = \Theta(g(n))$. But we have also shown that $n^n = \omega(n!)$ and $\{\omega, O\}$ are incompatible. So $2^{f(n)} = n^n \neq O(n!); n! = 2^{g(n)}$. So the implication is not true. The converse would be :$[2^{f(n)} = O(2^{g(n)}] \implies [f(n) = O(g(n)]$ and this is true as shown below:

   If $f(n) = O(g(n))$, then we show that $\lg(f(n)) = O(\lg(g(n)))$

   $f(n) \leq c_2 g(n) \implies \lg[f(n)] \leq \lg[c_2 g(n)] = \lg[g(n)] + \lg c_2 \leq 2\lg[g(n)]$ for large n.

3. Exercise 3.1-4:

   Solution:

   (i) Is $2^{n+1} = O(2^n)$: $\lim_{n\to\infty}[\frac{2^{n+1}}{2^n}] = 2$ and therefore the answer is yes.

   (ii) Is $2^{2n} = O(2^n)$: $\lim_{n\to\infty}[\frac{2^{2n}}{2^n}] = \infty$ and therefore the answer is no

4. Exercises 3.2-4: Are functions $\lceil \lg n \rceil!$ and $\lceil \lg\lg n \rceil!$ polynomially bounded?

   Solution:

   A function $f(n)$ is said to be polynomially bounded iff $f(n) = O(n^b)$ for some fixed b.

   Let $\lceil \lg n \rceil = k$. This is equivalent to $[k-1 < \lg n \leq k] \iff 2^{k-1} < n \leq 2^k$. $\lceil \lg n \rceil! = k!$

   $n^b \leq (2^k)^b = 2^{bk} < k!$ since $bk < k \lg k$ for large values of k. Hence, the first function is not polynomially bounded.

   Let $\lceil \lg\lg n \rceil = k \iff k - 1 < \lg\lg n \leq k \iff 2^{k-1} < \lg n \leq 2^k \iff 2^{2^{k-1}} < n \leq 2^{2^k}$

   $\lceil \lg\lg n \rceil! = k!$

   $n^b \leq (2^{2^k})^b = 2^{b2^k} > k!$ since $b2^k > k \lg k$ for large k. Hence, the second function is polynomially bounded.

5. Show that (i)$\sum_{i=1}^{i=n} i^2 = \Theta(n^3)$

Solution: We start with the relation:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=0}^{n} (i+1)^3 \\
&= \sum_{i=0}^{n} (i^3 + 3i^2 + 3i + 1) \\
&= \sum_{i=0}^{n-1} i^3 + 3\sum_{i=0}^{n-1} i^2 + 3\sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} 1
\end{aligned}$$

Hence

$$\sum_{i=0}^{n-1} i^2 = \frac{2n^3 - 3n^2 + n}{6}$$

Hence

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= \frac{2(n+1)^3 - 3(n+1)^2 + n + 1}{6} \\
&= \frac{2n^3 + 3n^2 + n}{6} \\
&= \frac{n(n+1)(2n+1)}{6} \\
&= \Theta(n^3)
\end{aligned}$$

This gives you the method of finding $\sum_{i=1}^{n} i^k$ for general values of k by induction.