Solution to Assignment #2:

1. (4.3-6) Show that \( t(n) = O(n \lg n) \) for the relation:

\[
    t(n) = 2t\left(\left\lfloor \frac{n}{2} \right\rfloor + 17\right) + n
\]

**Solution:** Let \( m = n - 34 \). This implies that \( \left\lfloor \frac{n}{2} \right\rfloor + 17 = \left\lfloor \frac{m}{2} \right\rfloor + 34 \). Hence our equation becomes:

\[
    t(m + 34) = 2t\left(\left\lfloor \frac{m}{2} \right\rfloor + 34\right) + (m + 34)
\]

\[
    s(m) = 2s\left(\left\lfloor \frac{m}{2} \right\rfloor \right) + \Theta(m)
\]

\[
    = \Theta(m \lg m)
\]

if we let \( S(m) = t(m + 34) \). Hence,

\[
    t(n) = t(m + 34) = s(m) = \Theta(m \lg m) = \Theta((n - 34) \lg(n - 34))
\]

(a) Now we show that \((n+c) \lg(n+c) = \Theta(n \lg n)\) and this will complete the proof.

\[
    \lim_{n \to \infty} \frac{(n+c) \lg(n+c) - n \lg n}{n \lg n}
\]

\[
    = \lim_{n \to \infty} \frac{(\lg(n+c) - \lg n)}{\lg n} + \lim_{n \to \infty} \frac{c \lg(n+c)}{n \lg n}
\]

\[
    = \lim_{n \to \infty} \frac{(\lg(1 + \frac{c}{n})}{\lg n} = 0
\]

Hence

\[
    \lim_{n \to \infty} \frac{(n+c) \lg(n+c)}{n \lg n} = 1
\]

and therefore, \((n+c) \lg(n+c) = \Theta(n \lg n)\).

(b) 4.3-9: Solve the relation \( t(n) = 3t(\sqrt{n}) + \log n \)

**Solution:**

Let \( \log n = m; n = 2^m \). The above equation changes to

\[
    t(2^m) = 3t(2^{\frac{m}{2}}) + m
\]

Now let \( t(2^x) = s(x) \) and the equations changes to

\[
    s(m) = 3s\left(\frac{m}{2}\right) + m
\]

This equation is solved by master theorem (case 1) to yield the solution \( s(m) = \Theta(m^{\log_2 3}) \). Hence the solution of the original equation is \( t(n) = \Theta((\log n)^{\log_2 3}) \).
2. **4.4-2**: Use a recursion tree to give an asymptotically tight solution to the relation:

\[ t(n) = t\left(\frac{n}{2}\right) + n^2 \]

Use substitution method to verify your answer.

\[
\begin{align*}
&n^2 \\
&\frac{(n/2)^2}{(n/4)^2} \\
&\frac{(n/2^k)^2}{\sum_{j=0}^{k} (1/2)^j} \\
&\text{where } k = \log n. \text{ Adding we get}
\end{align*}
\]

\[ t(n) = n^2 \sum_{j=0}^{k} \left(\frac{1}{2}\right)^j = \Theta(n^2). \]

**Verification**: By using induction hypothesis on smaller values we get

\[
\begin{align*}
t(n) &\leq n^2 + c\left(\frac{n}{2}\right)^2 \\
&= n^2\left[1 + \frac{c}{4}\right] \\
&\leq cn^2 \text{if } c > \frac{4}{3}
\end{align*}
\]

This shows that \( t(n) = O(n^2) \). By a similar analysis you can show that \( t(n) = \Omega(n^2) \).

**4.4-6**: Use a recursion tree to give an asymptotically tight solution to the relation:

\[ t(n) = t(\alpha n) + t((1 - \alpha)n) + cn \]
where \( c > 0 \), and \( 0 < \alpha < 1 \) are given constants.

\[
\begin{align*}
c_n(t(\alpha n)) &= c_n(t((1-\alpha)n)) \\
&= c \alpha^n (1-\alpha)^n \\
&= c \alpha^n (1-\alpha)^n \\
&= c \alpha^n (1-\alpha)^n \\
&= c_n(t(\alpha^2 n)) = c_n(t(\alpha^{(1-\alpha)n}) = c n
\end{align*}
\]

The depth of the tree is \( \frac{\lg n}{\lg \alpha} = \Theta(\lg n) \). Hence,

\[
t(n) = \Theta(n \lg n)
\]

Hence \( t(n) = \Omega(n \lg n) \).

3. Problem 4-3: (b),(c),(f)

b) \( t(n) = 3t(n/3) + \frac{n}{\lg n} \)

\( f(n) = \frac{n}{\lg n} \); \( n^{\log_3 a} = n \). So both cases 2 and 3 do not apply. Moreover, \( \lim_{n \to \infty} \frac{n^{1-\epsilon}}{f(n)} = 0 \) and hence \( f(n) = \omega(n^{1-\epsilon}) \). So we have \( f(n) \neq O(n^{\log_3 a - \epsilon}) \). So the master theorem does not apply in this case. This equation is of the form \( t(n) = at(n/3) + n^{\log_3 a}(\lg n)^k \); \( b > 1; a \geq 1 \) but \( k \geq 0 \). In this case \( k = -1 \). So we try iteration method after changing variables.

Let \( \log_3 n = k; n = 3^k; t(3^k) = s(k) \) and the equation becomes:

\[
t(n) = s(k) = 3s(k-1) + \frac{3^k}{ck}
\]

\[
= \frac{3^k}{ck} + 3 \frac{3^{k-1}}{c(k-1)} + \ldots
\]

\[
= \frac{1}{c} 3^k \sum_{i=1}^{k} \frac{1}{i}
\]

\[
= \Theta(3^k \lg k)
\]

\[
= \Theta(n \lg \lg n)
\]

Here \( \lg n = \frac{\log_3 n}{\log_2 3}; c = \frac{1}{\log_2 3} \).
c. \( t(n) = 4t(\frac{n}{3}) + n^2\sqrt{n} \)

\( f(n) = n^{2.5} = \Omega(n^{2+\varepsilon}) \); \( af(n) = 4(\frac{n}{3})^{2.5} = \frac{n^{2.5}}{\sqrt[3]{9}} \leq 0.8n^{2.5} \)

Hence this is case 3 of master theorem and so the result is \( t(n) = \Theta(n^{2.5}) \).

d. \( t(n) = t(\frac{n}{4}) + t(\frac{n}{2}) + t(\frac{n}{2}) + n \)

Substitution Method: Since \( \frac{1}{4} + \frac{1}{2} + \frac{1}{2} = \frac{7}{8} < 1 \), guess \( t(n) \leq cn \) and prove by induction. Using induction hypothesis

\[
    t(n) \leq \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \right) cn + n \\
    = \frac{7}{8} cn + n \\
    \leq cn \text{ for } c \geq 8
\]

Hence \( t(n) = O(n) \). But \( t(n) \geq n \) and hence \( t(n) = \Omega(n) \) and therefore \( t(n) = \Theta(n) \).

This completes the proof.

4. (4-4) Fibonacci Sequence:

\[
F_0 = 0 \quad F_1 = 1 \quad F_i = F_{i-1} + F_{i-2} \quad i \geq 2
\]

(a)

\[
F(z) = \sum_{i=0}^{\infty} F_i z^i = z + \sum_{i=2}^{\infty} (F_{i-1} + F_{i-2}) z^i \\
= z + z \sum_{i=0}^{\infty} F_i z^i + z^2 \sum_{i=0}^{\infty} F_i z^i \\
= z + zF(z) + z^2F(z) \\
= \frac{z}{1-z-z^2} \\
= \frac{1}{(1-\phi z)(1-\hat{\phi} z)}
\]

the last of these follows from the fact that \( \frac{1}{z} \) and \( \frac{1}{\hat{\phi} z} \) are the roots of \( [1 - z - z^2 = 0] \).

(b)

\[
\frac{z}{(1-\phi z)(1-\hat{\phi} z)} = \frac{1}{\sqrt{5}} \left( \frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z} \right) \\
\frac{1}{(1-\phi z)} = \sum_{i=0}^{\infty} \phi^i z^i \text{ and } \frac{1}{(1-\hat{\phi} z)} = \sum_{i=0}^{\infty} \hat{\phi}^i z^i. \text{ Hence}
\]
(c) \[
\mathcal{F}(z) = \frac{1}{\sqrt{5}} \sum_{i=0}^{\infty} (\phi^i - \hat{\phi}^i) z^i
\]

(d) Hence
\[
F_i = \frac{1}{\sqrt{5}} (\phi^i - \hat{\phi}^i)
\]
and \(|\hat{\phi}^i| < 1\). Hence \(F_i = |\frac{\phi^i}{\sqrt{5}}|\).

(e) Want to show that \(F_{i+2} \geq \phi^i\) for \(i \geq 0\)

We do this by induction. Clearly true for \(i = 0\). Suppose it is true for \(i \leq k\) Will show for \(i = k + 1\).

\[
F_{k+3} = F_{k+1} + F_{k+2} \geq \phi^{k-1} + \phi^k = \phi^{k+1} \left[ \frac{1}{\phi^2} + \frac{1}{\phi} \right] = \phi^{k+1} \left[ \frac{\phi+1}{\phi^2} \right] = \phi^{k+1}.
\]

\[
\frac{1}{\phi^2} = 1 - \frac{1}{\phi}
\]
\[
\phi^2 = 1 + \phi
\]
\[
\left( \frac{\phi+1}{\phi^2} \right) = 1
\]

(Recall \(\frac{1}{\phi}\) is a solution of the equation
\[
1 - z - z^2 = 0
\]
The result now follows.