Assignment #7

1. 34-1: (a),(b)

(a) The corresponding decision problem is the following: Given \( G = [V, E] \), and a positive integer \( k \), is there an independence set in \( G \) of size \( k \) is the decision problem. Thus

\[
\text{INDEPENDENCE} = \{(G, k) : G \text{ is a graph with an independence set of size } k\}
\]

To show that it is in NP, we use the independence set itself as the certificate. It is easy to verify that a given set is indeed an independent set or not and check its size in polynomial time.

To show that it is NP-complete we show that CLIQUE\( \leq_p \)INDEPENDENCE.

Given an instance of CLIQUE with \((G, k)\), let \( \overline{G} \) be the complement graph of \( G \). Let the corresponding INDEPENDENCE instance be on the \((\overline{G}, k)\). There is a clique of size \( k \) in graph \( G \) iff there is an independent set of size \( k \) in \( \overline{G} \). In fact the same set does it. Hence the reduction is complete.

(b) Run INDEPENDENCE over \( G \) with values of \( k \) in a binary search manner.

2. 34.5-2

To show that this problem is in NP, we use the vector \( x \) as the certificate. We now have to check that it satisfies each constraint in the system \( Ax \leq b \) and this can be done in polynomial time. To show that it is NP-complete, we show that 3-SAT\( \leq_p 0 - 1\)IP. To do this for each variable in 3-SAT we have a variable in 0 - 1IP and for each clause we have a constraint. For example, if a clause looks like \((x_3 \lor \neg x_5 \lor x_8)\) the corresponding constraint would be \(y_3 + (1 - y_5) + y_8 \geq 1\). The size of this IP is polynomially related to the boolean expression and the instance of 3-SAT has a yes answer iff each clause evaluates to 1. This happens iff the IP has a solution. The reduction is complete.

3. 34.5-5

Solution:

Certificate is \( A \) itself. Given \( A \) it is easy to check if \( A \) is the required set in polynomial time by adding numbers in \( A \) and \( S - A \).

To show that SET-PARTITION \( \in \) NPC: will show SUBSET-SUM\( \leq_p \)SET-PARTITION.

Let the instance of SUBSET-SUM be \((S, t)\). In this problem we want to know if \( \exists B \subseteq S : \sum_{x \in B} x = t \).

Let \( \tilde{S} = S \cup \{t+1\} \cup \{(\sum_{x \in S} x) - t + 1\} \) and consider the SET-PARTITION problem with \( \tilde{S} \).
If the answer to this partition problem is "yes" with $A \subseteq \tilde{S}$, then $\sum_{y \in A} y = \sum_{y \in \tilde{S} - A} y$. Exactly one of the last two elements in $\tilde{S}$ is in $A$ and the other is in $\tilde{S} - A$. This is because their sum exceeds the sum of all other elements (and all elements are positive). For the sake of specificity, let $\{t + 1\} \in A$. This implies $\{(\sum_{x \in S} x) - t + 1\} \in \tilde{S} - A$. Hence

$$\left( \sum_{y \in A - \{t+1\}} y \right) + (t + 1) = \left( \sum_{y \in \tilde{S} - A} y \right) + \left( \sum_{y \in S} y \right) - t + 1$$

$$\implies \left( \sum_{y \in \tilde{S} - A} y \right) = t$$

If the answer to SUBSET-SUM is yes, let $B \subseteq S$ satisfy the relation that $\sum_{y \in B} y = t$. Let $\tilde{S} - A$ in SET-PARTITION be $B \cup \{(\sum_{x \in S} x) - t + 1\}$ it is easy to show that this works. Thus, SUBSET-SUM $\leq_p$ SET-PARTITION. $t$ can be computed in polynomial time.

4. 34.1-6: do union, intersection, complementation, and concatenation.

Solution:
Complementation: Must show that $\{L \in P\} \implies \{\bar{L} \in P\}$

$L \in P$ implies that there is an algorithm $A$ that decides $L$ and the time taken by $A$ is given by $T(n) = O(n^k)$. This implies that there is a constant $c$ such that $T(n) \leq cn^k$. Now given a string $x \in \{0,1\}^*$ run algorithm $A$ till either it accepts or rejects $x$. If it accepts then output $x \notin \bar{L}$ else output $x \in \bar{L}$. Thus there is an algorithm $A'$ that decides $\bar{L}$ in polynomial time.

Rest is left to you.

5. 34.2-9

Solution:
To do this we must show $\{L \in P\} \implies \{\bar{L} \in NP\}$.

But $\{L \in P\} \implies \{\bar{L} \in P\}$ by 34.1-6. Also $P \subseteq NP$. So the result follows.