Assignment #7

1. 34-1: (a),(b)
   
   (a) The corresponding decision problem is the following: Given $G = [V, E]$, and a positive integer $k$, is there an independence set in $G$ of size $k$ is the decision problem. Thus
   
   $\text{INDEPENDENCE} = \{(G, k) : G \text{ is a graph with an independence set of size } k\}$
   
   To show that it is NP, we use the independence set itself as the certificate. It is easy to verify that a given set is indeed an independent set or not and check its size in polynomial time.
   
   To show that it is NP-complete we show that $\text{CLIQUE} \leq_p \text{INDEPENDENCE}$. Given an instance of CLIQUE with $(G, k)$, let $\overline{G}$ be the complement graph of $G$. Let the corresponding INDEPENDENCE instance be on the $(G, k)$. There is a clique of size $k$ in graph $G$ iff there is an independent set of size $k$ in $\overline{G}$. In fact the same set does it. Hence the reduction is complete.

   (b) Run INDEPENDENCE over $G$ with values of $k$ in a binary search manner.

2. 34.5-2

   To show that this problem is in NP, we use the vector $x$ as the certificate. We now have to check that it satisfies each constraint in the system $Ax \leq b$ and this can be done in polynomial time. To show that it is NP-complete, we show that $3\text{-SAT} \leq_p 0 - 1\text{IP}$. To do this for each variable in 3-SAT, we have a variable in $0 - 1\text{IP}$ and for each clause we have a constraint. For example, if a clause looks like $(x_3 \lor \neg x_5 \lor x_8)$ the corresponding constraint would be $y_3 + (1 - y_5) + y_8 \geq 1$. The size of this IP is polynomially related to the boolean expression and the instance of 3-SAT has a yes answer iff the IP has a solution. The reduction is complete.

3. 34.5-5: Show that $\text{SET-PARTITION} \in \text{NP-C}$

   **Solution:**
   
   Certificate is $A$ itself. Given $A$ it is easy to check if $A$ is the required set in polynomial time by adding numbers in $A$ and $S - A$.
   
   To show that $\text{SET-PARTITION} \in \text{NP-C}$: will show $\text{SUBSET-SUM} \leq_p \text{SET-PARTITION}$.
   
   Let the instance of SUBSET-SUM be $(S, t)$. In this problem we want to know if $\exists B \subseteq S : \sum_{x \in B} x = t$.
   
   Let $\hat{S} = S \cup \{t+1\} \cup \{(\sum_{x \in S} x) - t + 1\}$ and consider the SET-PARTITION problem with $\hat{S}$.
   
   If the answer to this partition problem is "yes" with $A \subseteq \hat{S}$, then $\sum_{y \in A} y = \sum_{y \in \hat{S} - A} y$. Exactly one of the last two elements in $\hat{S}$ is in $A$ and the other
is in $\tilde{S} - A$. This is because their sum exceeds the sum of all other elements (and all elements are positive). For the sake of specificity, let \( \{t + 1\} \in A \). This implies \( \{\sum_{x \in S} x - t + 1\} \in \tilde{S} - A \). Hence

\[
\left( \sum_{y \in A - \{t + 1\}} y \right) + (t + 1) = \left( \sum_{y \in \tilde{S} - A} y \right) + (\sum_{y \in S} y) - t + 1
\]

\[
\Rightarrow \left( \sum_{y \in \tilde{S} - A} y \right) = t
\]

If the answer to SUBSET-SUM is yes, let $B \subseteq S$ satisfy the relation that $\sum_{y \in B} y = t$. Let $\tilde{S} - A$ in SET-PARTITION be $B \cup \{\sum_{x \in S} x - t + 1\}$ it is easy to show that this works. Thus, SUBSET-SUM $\leq_p$ SET-PARTITION. $t$ can be computed in polynomial time.

4. 34.1-6: do union, intersection, complementation, and concatenation.

**Solution:**

Complementation: Must show that $\{L \in P\} \implies \{\overline{L} \in P\}$

$L \in P$ implies that there is an algorithm $A$ that decides $L$ and the time taken by $A$ is given by $T(n) = O(n^k)$. This implies that there is a constant $c$ such that $T(n) \leq cn^k$. Now given a string $x \in \{0, 1\}^*$ run algorithm $A$ till either it accepts or rejects $x$. If it accepts then output $x \not\in \overline{L}$ else output $x \in \overline{L}$. Thus there is an algorithm $A'$ that decides $\overline{L}$ in polynomial time.

Rest is left to you.

5. 34.2-9

**Solution:**

To do this we must show $\{L \in P\} \implies \{\overline{L} \in NP\}$.

But $\{L \in P\} \implies \{\overline{L} \in P\}$ by 34.1-6. Also $P \subseteq NP$. So the result follows.