Robust and Adaptive Spanning Trees

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In all problems considered in this paper, the input is the same: An undirected connected graph $G = [V; E]$ with edge weights $w : E \to R$. We will assume that G is a complete graph on V without loss.

In the *Robust Spanning Tree* problem, the scenario is as follows: First one selects a spanning tree T in the graph G; then an adversary removes a node [an edge] from T. We want to select the tree maximize the sum of the weights of the remaining edges.

In the *Adaptive Spanning Tree* problem, the scenario is modified as follows: First one selects a spanning tree T in the graph G; then an adversary removes a node [an edge] from T. The first person now adds edges from those not selected or removed so as to connect the components of the tree that remain. We want to minimize the total weight of the original tree plus the patch up cost.

There is an alternative way to define the Adaptive Spanning tree problem in the following manner: First one selects a spanning tree T_1 in the graph G; then an adversary removes one (or more) node(s) [edge(s)] from T_1. Now the first person selects another spanning tree T_2 in the remaining subgraph. We want to minimize the sum of weights of edges in the set $T_1 \cup T_2$.

One of the questions we consider is: are the above two formulations equivalent?

0.0.1 Robust Optimization with edge(s) removal

It is "clear" that the adversary will remove the edge(s) with largest weight. This problem is the following problem for the case where one edge is removed:

$$\max_{T \in \Upsilon} [w(T) - \max_{e \in T} w(e)]$$

where $w(T) = \sum_{e \in T} w(e)$ and Υ is the set of all spanning trees in G. The following lemma solves the first player's problem:

Lemma 1 *Any maximum weight spanning tree in G is optimal.*

Proof. Let T_1 be a maximum weight spanning tree and let T_2 be any other spanning tree of G. Let the weights of edges in both trees be ordered in decreasing order and let the vectors of weights so ordered be $w(T_1) = [w_1 \geq w_2 \geq ... \geq w_m]$ and $w(T_2) = [w'_1 \geq w'_2 \geq ... \geq w'_m]$. It is well known that $w(T_1) \geq w(T_2)$ [see

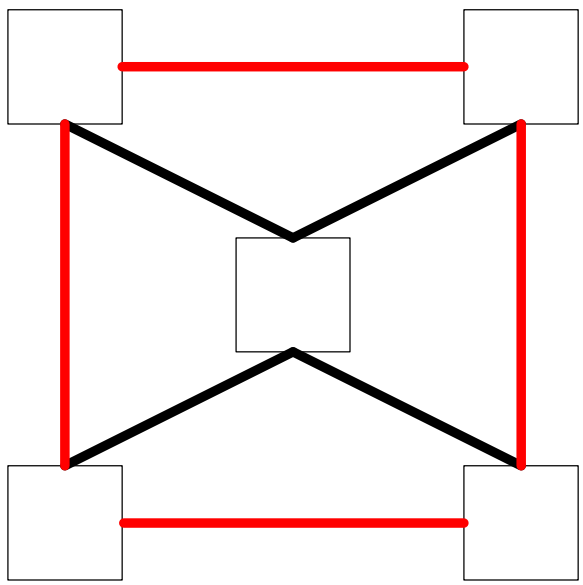
FF]. Hence, removal of the largest k edges from each tree yields a better solution in case for T_1 for all values of $k \leq m$. ■

Remark 2 *This result holds when we consider an arbitrary matroid.*

Remark 3 *But the situation is quite different even when one node is removed. [There is no concept equivalent to a node in general matroid.]*

0.0.2 Robust Optimization with node removal

Here is an example to show now the solution may not be a maximum weight spanning tree.



Weights of red edges is 1 and those of black edges is $1+\epsilon$ where $0 < \epsilon$. Maximum weight spanning tree consists of black edges with total weight equal to $4(1+\epsilon)$; but if the central node is removed we will be left with a value of 0 for the remaining edges. Consider a spanning tree consisting of three red edges and one black edge forming a hamiltonian path. The weight is $4+\epsilon$; maximum loss can be at most $2+\epsilon$ and hence the weight of remaining

edges is $2 > 0$. Hence best solution is not the maximum weight spanning tree for this problem. Moreover, this problem includeds as a special case the problem of finding a spanning tree in an undirected graph whose maximum degree at a node is minimum and this problem is known to be NP-hard since it generalizes the hamiltonian path problem. We show this now.

Let G be a graph with all edge weights equal to 1. For any spanning tree, it should be clear that the adversary will remove the node with the maximum degree resulting in a loss equal to the degree of that node. Hence the tree that is

optimal for the robust problem will be that which minimizes the largest degree. Hence the result follows.

Hence we need to look for approximation algorithms for this problem along the lines of finding approximation algorithms for minimum degree spanning tree problem and its generalizations.

0.0.3 Adaptive Optimization with one edge removal and its patching:

Here is the adaptive optimization problem (using the first form) with one edge removal formulated:

$$\min_{T\in\Upsilon}\{w(T)+\max_{e\in T}[\min_{f\neq e:\{T-e+f\}\in\Upsilon} w(f)]\}$$

Theorem 4 *Let T_1 be a minimum weight spanning tree of G. Then T_1 is optimal for the above problem and hence:*

$$w(T_1)+\max_{e\in T_1}[\min_{f\neq e:\{T_1-e+f\}\in\Upsilon} w(f)] = \min_{T\in\Upsilon}\{w(T)+\max_{e\in T}[\min_{f\neq e:\{T-e+f\}\in\Upsilon} w(f)]\}$$

We need a few results before proving this theorem some of which are well known.

Lemma 5 *Let T_1 be a minimum weight spanning tree. Let $[S,\bar{S}]$ be any cut that partitions the set of nodes V of G into two nonempty disjoint sets. Let w be the minimum weight of any edge that crosses this cut. Then, among the edges of T_1 that cross this cut, at least one of them has a weight equal to w.*

Proof. We give a proof for the sake of completeness (although this is probably a well known result). Suppose the result is not true. Let e be a minimum weight edge crossing the cut and by supposition, $e \notin T_1$. Adding e to T_1 creates a unique cycle C of which e is a part. Since $e \in C$ crosses the cut, there is another edge f that also crosses the cut but whose edge weight $w(f) > w(e)$ by supposition. Consider the spanning tree $T_2 = T_1 - f + e$. It should be clear that $w(T_2) < w(T_1)$ and this contradicts the assumption that T_1 is a minimum weight spanning tree of G. ■

Proof. (of Main Theorem): Suppose the claim in the theorem is not true. Let $T_2 \neq T_1$ be a spanning tree that satisfies the relation:

$$w(T_2)+\max_{e\in T_2}[\min_{f\neq e:\{T_2-e+f\}\in\Upsilon} w(f)] = \min_{T\in\Upsilon}\{w(T)+\max_{e\in T}[\min_{f\neq e:\{T-e+f\}\in\Upsilon} w(f)]\}$$

Let $(u,v) = e_1 \in T_2\backslash T_1$. Consider $T_2\backslash e_1$; this consists of two subtrees one of which contains an edge incident at u and the other contains an edge incident at v. Let these two subtrees (that are part of T_2) be denoted respectively by T_2^u and T_2^v. Let

$$w(f_1) = [\min_{f\neq e_1:\{T_2-e_1+f\}\in\Upsilon} w(f)]$$

We will call the edge f_1 as a *minimum weight patching edge* for e_1 with respect to tree T_2 and denote f_1 by $patch_{T_2}(e_1)$. Similar remarks apply to any edge of any spanning tree.

Remark 6 *Minimum Weight Patching edges may not be unique but their weights are equal.*

■

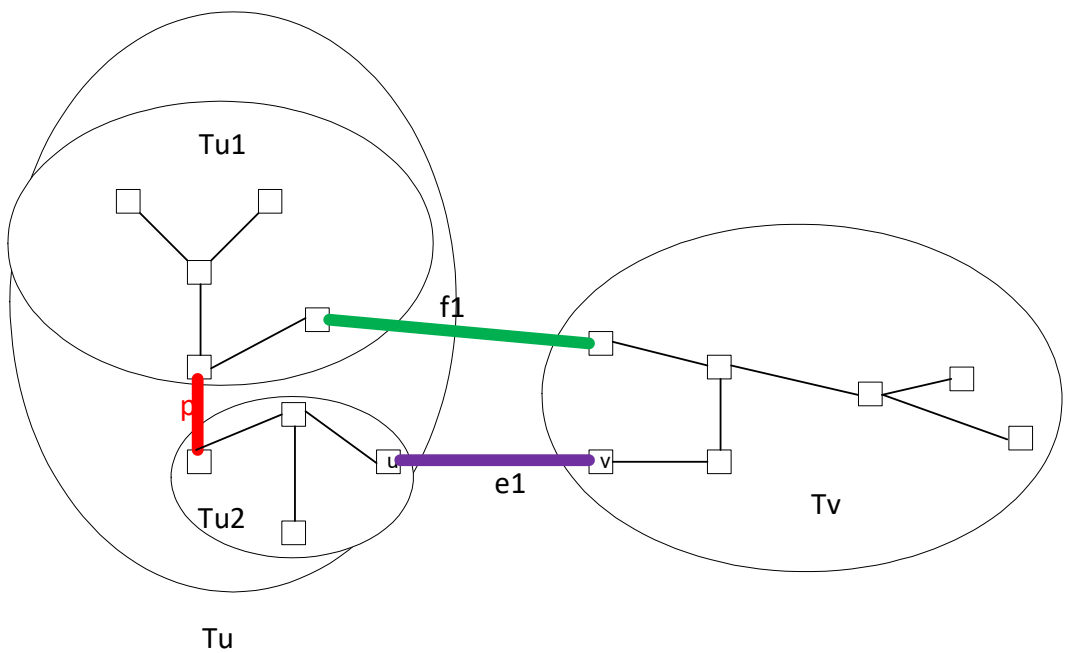
Proof. (continued): The edge f_1 connects the subtrees T_2^u and T_2^v in place of the edge e_1. The choice would be some minimum weight edge not in T_2 that crosses the cut with $S = \{\text{set of nodes in } T_2^u\}$. Among the edges of T_1 that cross this cut [and there are such edges none of which is e_1], we select one with minimum weight. This also happens to be minimum weight edge that crosses this cut by above lemma. Hence $w(f_1) \leq w(e_1)$. Consider the tree $T_3 = T_2 - e_1 + f_1$. This tree T_3 has one more edge in common with tree T_1 than tree T_2. Weight of the minimum weight patching edge for f_1 with respect to tree T_3 can not be larger than $w(e_1)$ since e_1 is a candidate for patching for f_1 with respect to tree T_3. Moreover,

$$w(T_3)+w(patch_{T_3}(f_1)) \leq w(T_3)+w(e_1) = w(T_2)+w(f_1) = w(T_2)+w(patch_{T_2}(e_1))$$

Now if we show that

$$w(T_3)+w(patch_{T_3}(p)) \leq \max[w(T_3)+w(e_1); w(T_2)+w(patch_{T_2}(p))] \qquad \forall p \in T_2 \backslash e_1$$

then it would show that T_3 is no worse than T_2 for the adaptive spanning tree problem. By repeating this process the thereom would be proved since T_3 has one more common edge with tree T_1. Without loss we will show this for some edge p in T_2^u. Removing p from T_2^u breaks it into two parts $T_2^u(1)$ and $T_2^u(2)$. See diagram below:



$$\text{Tu} \quad = \quad T_2^u;\ \text{Tv} = T_2^v; \text{Tu1} = T_2^u(1);\ \text{Tu2} = T_2^u(2);\ \text{e1} = e_1; \text{f1} = f_1$$

There are two cases to consider: (i)one end of f_1 is in $T_2^u(1)$ [this is what is shown in the above diagram]; (ii)one end of f_1 is in $T_2^u(2)$ (this is shown in diagram below). In the first case, when we remove edge p from T_3 we get two subtrees that are respectively $T_2^u(2)$ and $T_2^u(1) + f_1 + T_2^v$. In the second case we get two subtrees that are respectively $T_2^u(1)$ and $T_2^u(2) + f_1 + T_2^v$.

If $patch_{T_2}(p) = q$ does not cross the cut $(S, \bar{S})$ with $S = \{$set of nodes in $T_2^u\}$, then q is still a candidate for $patch_{T_3}(p)$ and hence

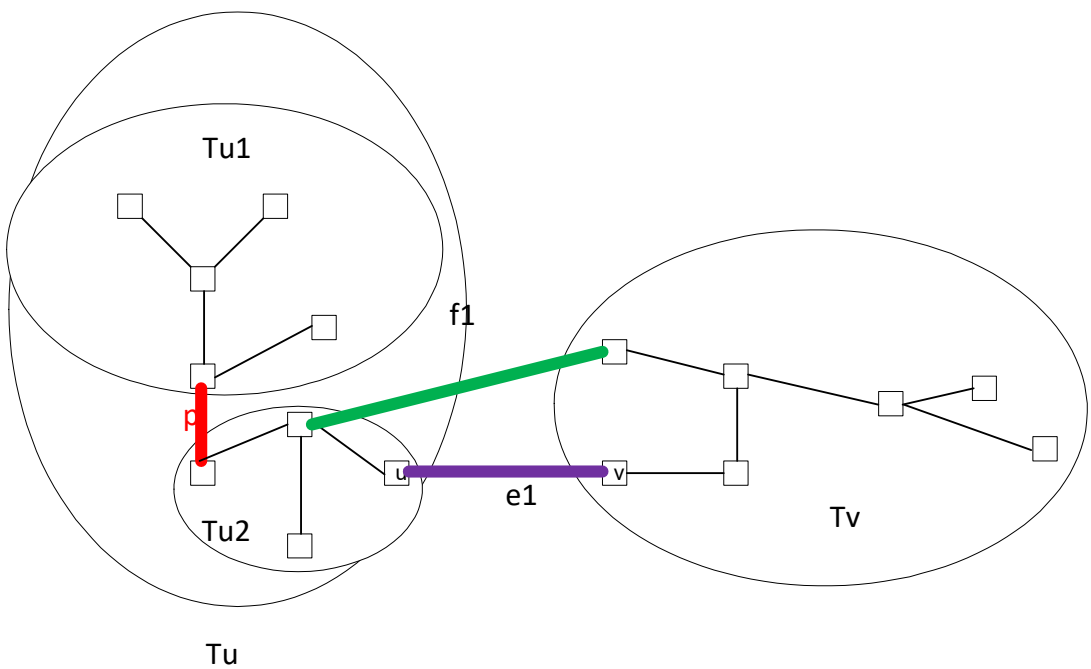
$$w(T_3) + w(patch_{T_3}(p)) \leq \max[w(T_3) + w(e_1); w(T_2) + w(patch_{T_2}(p))]$$

continues to hold. Now we assume that $patch_{T_2}(p) = q$ crosses the cut $(S, \bar{S})$. Consider the two cases separately:

Case (i): In this case the edge e_1 is a candidate for $patch_{T_3}(p)$ and hence

$$w(T_3) + w(patch_{T_3}(p)) \leq \max[w(T_3) + w(e_1); w(T_2) + w(patch_{T_2}(p))]$$

case (ii): See diagram below:



In this case $patch_{T_2}(p)$ is still a candidate for $patch_{T_3}(p)$ and hence

$$w(T_3) + w(patch_{T_3}(p)) \leq \max[w(T_3) + w(e_1); w(T_2) + w(patch_{T_2}(p))]$$

This completes the proof of the main theorem for one edge removal. ■

0.0.4 Adaptive Optimization with one node removal:

Let G be a graph with all edge weights equal to 1. For any spanning tree, it should be clear that the adversary will remove the node with the maximum degree resulting in additional cost of patching equal to the degree of that node. Hence the tree that is optimal for the adaptive optimization problem will be that which minimizes the largest degree. Hence the result follows that this problem is also NP-hard.